ME 380: Kinematics and Machine Dynamics
Loop Equations and Midterm Worksheet

In this worksheet, we are going to solve for the position-level kinematics of the Stephenson linkage, depicted in Figure 1.

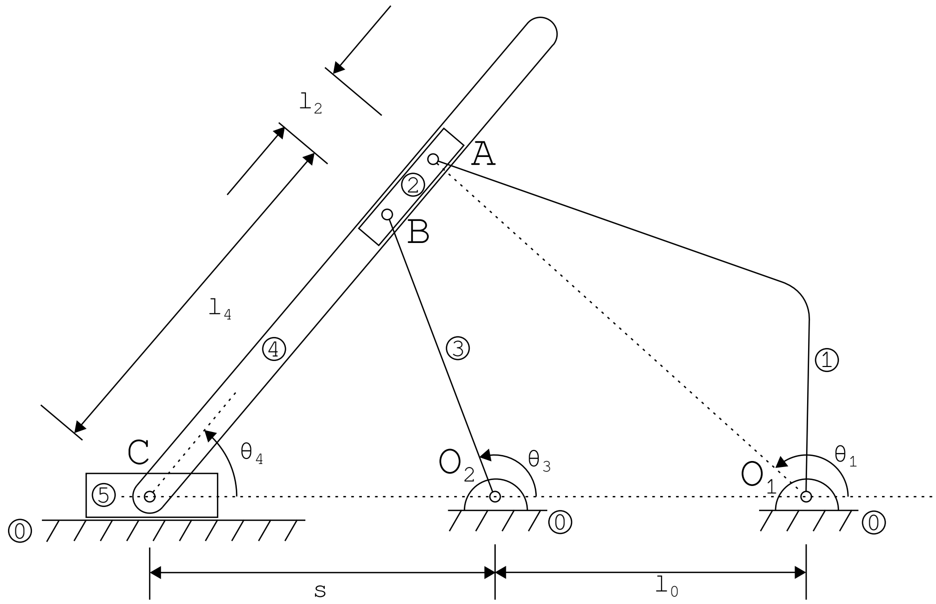

Figure 1: `Stephenson Linkage`

1. Determine the degrees of freedom of this mechanism. From this computation, decide how many variables you have to solve for in position-level kinematics.

> **Solution:** This is a planar mechanism with 5 moving links, 5 revolute and 2 prismatic joints. Therefore,
>
> $$\text{DoF } = 3 \cdot 5 - 2 \cdot 5 - 2 \cdot 2 = 1.$$

2. Based on your answer, decide how many loop equations you need, and write down these loop equations in *vector-form* (this means, do not worry about their representation in a particular coordinate frame yet).

> **Solution:** Let $p_0 = \overrightarrow{O_2O_1}$, $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AB}$, $p_3 = \overrightarrow{O_2B}$, $p_4 = \overrightarrow{CB}$, $p_5 = \overrightarrow{CO_2}$, and $p = \overrightarrow{B_4B_3}$, where B_4 and B_3 are points fixed on frames Σ_4 and Σ_3, respectively, coincident with B at the instant in consideration.

Remark *Notice that, in this construction $p = 0$. There is no reason to include this vector in your loop equations if you are only interested in solving the position-level kinematics. However, this will be crucial in our velocity- and acceleration-level analyses. Hence, we are including this vector for future reference.*

Two loop equations may then be written as

$$\begin{aligned} p_0 + p_1 + p_2 - p_3 &= 0, \\ p_3 - p - p_4 + p_5 &= 0. \end{aligned} \tag{1}$$

3. Assign coordinate frames Σ_i to each link i and derive the rotation matrices of each of these frames with respect to the world (inertial frame, Σ_0), i.e., find $\{R_i^0\}_1^5$.

Solution: We define the coordinate frames by choosing the common z-axes to point outside the page and the x-axes as follows

Σ_1: x_1-axis is the unit vector along p_1,

Σ_2: x_2-axis is the unit vector along p_4,

Σ_3: x_3-axis is the unit vector along p_3,

Σ_4: $x_4 = x_2$,

Σ_5: $x_5 = x_0$.

We can then identify the rotation matrices by $R_1^0 = R_{z,\theta_1}$, $R_2^0 = R_4^0 = R_{z,\theta_4}$, $R_3^0 = R_{z,\theta_3}$, and $R_5^0 = I$.

4. Express the vector loop equation you wrote down in question 2 in the inertial frame, Σ_0.

Solution: We have the following decompositions: $p_0 = l_0 x_0$, $p_1 = l_1 x_1$, $p_2 = -l_2 x_2$, $p_3 = l_3 x_3$, $p_4 = l_4 x_4$, and $p_5 = s x_5$. Notice that

$$x_i^m = R_k^m x_i^k.$$

This equation says that if we know the expression of the vector x_i in the frame Σ_k, we can find its expression in the frame Σ_m by multiplying x_i^k on the left by R_k^m. Now, apply this formula by inserting $m = 0$.

Hence, with the right-hand vectors expressed in the frame Σ_0, we have

$$x_1^0 = \begin{bmatrix}\cos\theta_1\\ \sin\theta_1\\ 0\end{bmatrix}, \; x_2^0 = \begin{bmatrix}\cos\theta_4\\ \sin\theta_4\\ 0\end{bmatrix}, \; x_3^0 = \begin{bmatrix}\cos\theta_3\\ \sin\theta_3\\ 0\end{bmatrix}.$$

Using these decompositions, we can write the equations (1) in the inertial frame as

$$\begin{aligned} l_0 + l_1\cos\theta_1 - l_2\cos\theta_4 - l_3\cos\theta_3 &= 0,\\ l_1\sin\theta_1 - l_2\sin\theta_4 - l_3\sin\theta_3 &= 0,\\ l_3\cos\theta_3 - l_4\cos\theta_4 + s &= 0,\\ l_3\sin\theta_3 - l_4\sin\theta_4 &= 0. \end{aligned} \tag{2}$$

5. (Home exercise) Suppose the link lengths are given as follows: $l_0 = |O_2O_1| = \frac{3}{5}$, $l_1 = |O_1A| = 1.0$, $l_2 = |AB| = \frac{1}{4}$, $l_3 = |O_2B| = \frac{7}{10}$. Use your Newton-Raphson algorithm to solve for

   (a) θ_3, θ_4, s and l_4 if $\theta_1 = \frac{4\pi}{5}$.

   Solution: Forward kinematics
   $(\theta_1 = 144°) : \theta_3 = 130.334°,\ \theta_4 = 12.519°,\ s = 2.856,\ l_4 = 2.462.$

   (b) θ_1, θ_3, θ_4, and l_4 if $s = 2.856$.

   Solution: Inverse kinematics
   $(s = 2.856) : \theta_1 = 144°,\ \theta_3 = 130.334°,\ \theta_4 = 12.519°,\ l_4 = 2.462$