

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name: _____

1. [30 points] This question concerns linear algebra.

(a) [5 points] Suppose x_k is the fraction of BSU students who prefer calculus to linear algebra at year k . The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k+1$, $\frac{1}{5}$ of those who prefer calculus change their mind. Also at year $k+1$, $\frac{1}{10}$ of those who prefer linear algebra change their mind.

Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

(b) [5 points] Give a **basis** for the space of 3 by 3 *diagonal matrices*.

Any 3x3 diagonal matrix has the form
 $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ so a basis is given by
 $\{e_1, e_2, e_3\}$ where $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) [5 points] Suppose A is the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Describe the **column space** of this particular matrix A . "All combinations of the four columns" is not a sufficient answer.

The column space of A is given by all combinations of the pivot columns:

$$C(A) = \left\{ v \in \mathbb{R}^3 : v = c_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix}, \forall c_1, c_2 \in \mathbb{R} \right\}.$$

- (d) [5 points] Consider the matrix

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 2 & -3 \\ 0 & -3 & 5 \end{bmatrix}$$

Find the **determinant** of the matrix A . Suppose $Ax = y$, for some $y \in \mathbb{R}^3$. Does there exist $x \in \mathbb{R}^3$ that solves this equation? Briefly explain your answer.

$$\det(A) = 0.$$

There does not exist $x \in \mathbb{R}^3$ for any y
 b/c the column space does not span $\mathbb{R}^3$.
 For some y (of measure zero), there does
 exist such x .

- (e) [5 points] We say that a vector space V is an **inner product space** if there is a binary function $f : V \times V \rightarrow \mathbb{R}$, called the *inner product*, defined on V . The *dot product* you have learned in your Calculus courses is an example of such a product whenever $V = \mathbb{R}^n$. For example, let $v = [v_1 \ v_2 \ v_3]^\top \in \mathbb{R}^3$ and $w = [w_1 \ w_2 \ w_3]^\top \in \mathbb{R}^3$. An inner product of v and w is given by

$$f(v, w) = v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Suppose V is an inner product space, and you are given two vectors, $v, w \in V$. Compute the angle between these two vectors.

$$\cos(\theta) = f(v, w) \Rightarrow \theta = \arccos \circ f(v, w).$$

- (f) [5 points] Suppose v_1 and v_2 are two vectors in an inner product space V . We say that v_1 and v_2 are **orthogonal** if the inner product of v_1 and v_2 vanishes (equals zero). Suppose a and b are two vectors in $\mathbb{R}^3$, which are linearly independent. Assume $\mathbb{R}^3$ is equipped with the inner product described in part (f). Find a vector in $\mathbb{R}^3$ that is orthogonal to *both* a and b .

$$c = a \times b.$$

- (g) [10 points (bonus)] Find the matrix P that **projects** every vector b in $\mathbb{R}^3$ onto the line in the direction of $a = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$.

Recall that a matrix P is a projection matrix onto a if and only if

1. $P^2 = P$
2. $Pa = a$
3. Any vector b in $\mathbb{R}^3$ can be decomposed as $b = \lambda a + c$ where $\lambda \in \mathbb{R}$ and $Pc = 0$.

$$P = \begin{pmatrix} | & | & | \\ p_1 & p_2 & p_3 \\ | & | & | \end{pmatrix}$$

$$\tilde{p}_1 = (e_1 \cdot a)a = \begin{pmatrix} 4 & 2 & 6 \end{pmatrix}^T$$

$$\tilde{p}_2 = (e_2 \cdot a)a = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix}^T$$

$$\tilde{p}_3 = (e_3 \cdot a)a = \begin{pmatrix} 6 & 3 & 9 \end{pmatrix}^T$$

$$p_i = \frac{\tilde{p}_i}{\|a\|^2}$$

2. [30 points] This question concerns statics and dynamics.
- (a) [10 points] Suppose a particle of mass m is undergoing a circular motion of radius l . What are the inertial (fictitious) forces on it?
- (b) [20 points] Consider the simple pendulum depicted in the Figure 1 below. The pendulum consists of a cable and a bob, connected to the world via a revolute joint. Assuming that the mass m is concentrated on the bob, the length of the cable is l and the gravity acts in the negative y -direction, draw the free-body diagram of the pendulum to find the equations of motion governing this system.

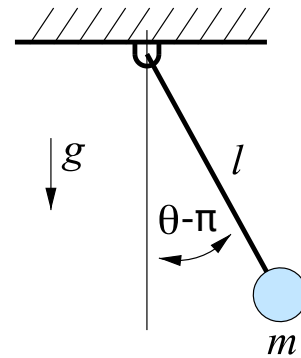


Figure 1: Simple Pendulum