Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Name/ID: ______________________________

1. [30 points] This question concerns linear algebra.

   (a) [5 points] Suppose x_k is the fraction of BSU students who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

   At year $k+1$, $\frac{1}{5}$ of those who prefer calculus change their mind. Also at year $k+1$, $\frac{1}{10}$ of those who prefer linear algebra change their mind.

   Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$

   (b) [5 points] Give a **basis** for the space of 3 by 3 *diagonal matrices.*

   (c) [5 points] Suppose A is the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

   Describe the **column space** of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

(d) [5 points] Consider the matrix

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 2 & -3 \\ 0 & -3 & 5 \end{bmatrix}$$

Find the **determinant** of the matrix A. Suppose $Ax = y$, for some $y \in \mathbb{R}^3$. Does there exist $x \in \mathbb{R}^3$ that solves this equation? Briefly explain your answer.

(e) [5 points] We say that a vector space V is an **inner product space** if there is a binary function $f : V \times V \to \mathbb{R}$, called the *inner product*, defined on V. The *dot product* you have learned in your Calculus courses is an example of such a product whenever $V = \mathbb{R}^n$. For example, let $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top \in \mathbb{R}^3$ and $w = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^\top \in \mathbb{R}^3$. An inner product of v and w is given by

$$f(v, w) = v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Suppose V is an inner product space, and you are given two vectors, $v, w \in V$. Compute the angle between these two vectors.

(f) [5 points] Suppose v_1 and v_2 are two vectors in an inner product space V. We say that v_1 and v_2 are **orthogonal** if the inner product of v_1 and v_2 vanishes (equals zero). Suppose a and b are two vectors in $\mathbb{R}^3$, which are linearly independent. Assume $\mathbb{R}^3$ is equipped with the inner product described in part (f). Find a vector in $\mathbb{R}^3$ that is orthogonal to *both* a and b.

(g) [10 points (bonus)] Find the matrix P that **projects** every vector b in $\mathbb{R}^3$ onto the line in the direction of $a = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^\top$.

Recall that a matrix P is a projection matrix onto a if and only if

1. $P^2 = P$
2. $Pa = a$
3. Any vector b in $\mathbb{R}^3$ can be decomposed as $b = \lambda a + c$ where $\lambda \in \mathbb{R}$ and $Pc = 0$.

2. [30 points] This question concerns statics and dynamics.

   (a) [10 points] Suppose a particle of mass m is undergoing a circular motion of radius l. What are the inertial (fictitious) forces on it?

   (b) [20 points] Consider the simple pendulum depicted in the Figure 1 below. The pendulum consists of a cable and a bob, connected to the world via a revolute joint. Assuming that the mass m is concentrated on the bob, the length of the cable is l and the gravity acts in the negative y-direction, draw the free-body diagram of the pendulum to find the equations of motion governing this system.

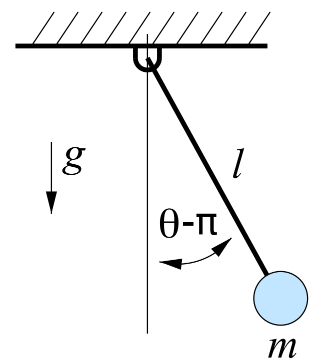

Figure 1: `Simple Pendulum`