1. Using the fact that $v_1 \cdot v_2 = v_1^T v_2$, show that the dot product of two free vectors does not depend on the choice of frames in which their coordinates are defined.

2. Show that the length of a free vector is not changed by rotation, i.e., that $\|v\| = \|Rv\|$.

3. Show that the distance between points is not changed by rotation i.e., that $\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$.

4. If a matrix R satisfies $R^T R = I$, show that the column vectors of R are of unit length and mutually perpendicular.

5. If a matrix R satisfies $R^T R = I$, then
a) show that $\det R = \pm 1$
b) Show that $\det R = \pm 1$ if we restrict ourselves to right-handed coordinate systems.

6. Verify Equations (2.3)-(2.5).

7. A **group** is a set X together with an operation $*$ defined on that set such that

   - $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
   - $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
   - There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$.
   - For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$.

   Show that SO(n) with the operation of matrix multiplication is a group.

8. Derive Equations (2.6) and (2.7).

9. Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

10. Consider the following sequence of rotations:

    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the current z-axis.
    (c) Rotate by ψ about the world y-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

11. Consider the following sequence of rotations:

    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the world z-axis.
    (c) Rotate by ψ about the current x-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

12. Consider the following sequence of rotations:

    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the current z-axis.
    (c) Rotate by ψ about the current x-axis.
    (d) Rotate by α about the world z-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

13. Consider the following sequence of rotations:

    (a) Rotate by ϕ about the world x-axis.
    (b) Rotate by θ about the world z-axis.
    (c) Rotate by ψ about the current x-axis.
    (d) Rotate by α about the world z-axis.

    Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

14. Find the rotation matrix representing a roll of $\frac{\pi}{4}$ followed by a yaw of $\frac{\pi}{2}$ followed by a pitch of $\frac{\pi}{2}$.

15. If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

16. Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$ and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}; R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

    Find the matrix R_3^2.

17. Verify Equation (2.46).

18. If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

19. Let $k = \frac{1}{\sqrt{3}}(1,1,1)^T$, $\theta = 90°$. Find $R_{k,\theta}$.

20. Show by direct calculation that $R_{k,\theta}$ given by Equation (2.46) is equal to R given by Equation (2.51) if θ and k are given by Equations (2.52) and (2.53), respectively.

21. Compute the rotation matrix given by the product

$$R_{x,\theta} R_{y,\phi} R_{z,\pi} R_{y,-\phi} R_{x,-\theta}$$

22. Suppose R represents a rotation of $90°$ about y_0 followed by a rotation of $45°$ about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

23. Find the rotation matrix corresponding to the set of Euler angles $\left\{\frac{\pi}{2}, 0, \frac{\pi}{4}\right\}$. What is the direction of the x_1 axis relative to the base frame?

24. Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

25. Unit magnitude complex numbers (i.e., $a + ib$ such that $a^2 + b^2 = 1$) can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

26. Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

27. Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned} -1 &= i^2 = j^2 = k^2, \\ i &= jk = -kj, \\ j &= ki = -ik, \\ k &= ij = -ji \end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = (n_x, n_y, n_z)^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, i.e., that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

28. Using $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$, and the results from Section 2.5.3, determine the rotation matrix R that corresponds to the rotation represented by the quaternion (q_0, q_1, q_2, q_3).

29. Determine the quaternion Q that represents the same rotation as given by the rotation matrix R.

30. The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $= (q_1, q_2, q_3)^T$. Show that the product of two quaternions, $Z = XY$ is given by

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

Hint: perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

31. Show that $Q_I = (1, 0, 0, 0)$ is the identity element for unit quaternion multiplication, i.e., that $QQ_I = Q_I Q = Q$ for any unit quaternion Q.

32. The conjugate Q^* of the quaternion Q is defined as

$$Q^* = (q_0, -q_1, -q_2, -q_3)$$

Show that Q^* is the inverse of Q, i.e., that $Q^*Q = QQ^* = (1, 0, 0, 0)$.

33. Let v be a vector whose coordinates are given by $(v_x, v_y, v_z)^T$. If the quaternion Q represents a rotation, show that the new, rotated coordinates of v are given by $Q(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component.

34. Let the point p be rigidly attached to the end effector coordinate frame with local coordinates (x, y, z). If Q specifies the orientation of the end effector frame with respect to the base frame, and T is the vector from the base frame to the origin of the end effector frame, show that the coordinates of p with respect to the base frame are given by

$$Q(0, x, y, z)Q^* + T \tag{2.73}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component.

35. Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin O_1 with respect to the original frame in each case?

36. Consider the diagram of Figure 2.15. Find the homogeneous transfor-

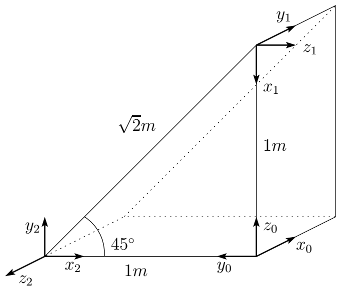


Fig. 2.15 Diagram for Problem 2-36.

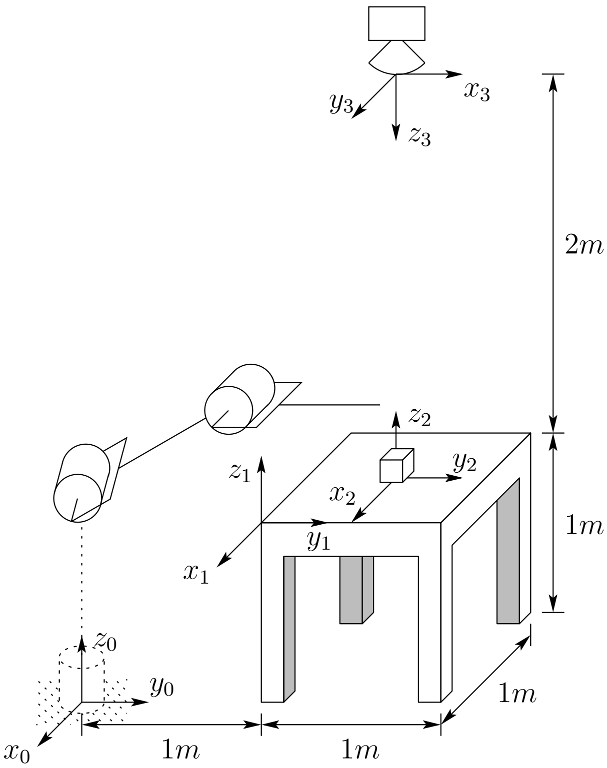


Fig. 2.16 Diagram for Problem 2-37.

mations H_1^0, H_2^0, H_2^1 representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0, H_2^1$.

37. Consider the diagram of Figure 2.16. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a

side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2m above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

38. In Problem 37, suppose that, after the camera is calibrated, it is rotated 90° about z_3. Recompute the above coordinate transformations.

39. If the block on the table is rotated 90° about z_2 and moved so that its center has coordinates $(0, .8, .1)^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.

40. Consult an astronomy book to learn the basic details of the Earth's rotation about the sun and about its own axis. Define for the Earth a local coordinate frame whose z-axis is the Earth's axis of rotation. Define $t = 0$ to be the exact moment of the summer solstice, and the global reference frame to be coincident with the Earth's frame at time $t = 0$. Give an expression $R(t)$ for the rotation matrix that represents the instantaneous orientation of the earth at time t. Determine as a function of time the homogeneous transformation that specifies the Earth's frame with respect to the global reference frame.

41. In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product

$$H = Rot_{x,\alpha}\text{Trans}_{x,b}\text{Trans}_{z,d}Rot_{z,\theta}$$

Determine which pairs of the four matrices on the right hand side commute. Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix, H.

RBM Exercise Solutions

(9) Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SO(2)$. Then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{T}$$

$$\Rightarrow \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

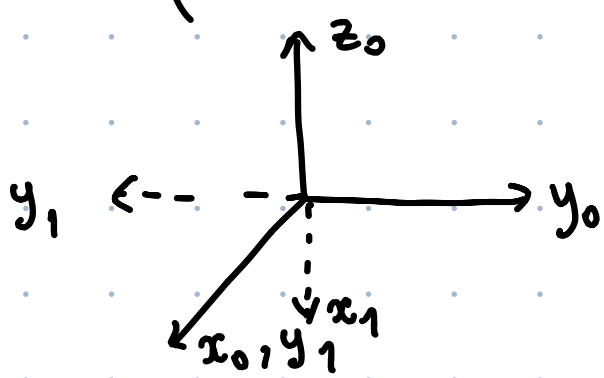
$$\Rightarrow \quad a_{11} = a_{22} \quad \text{and} \quad a_{12} = -a_{21}.$$

Define $\tan(\theta) = \dfrac{a_{21}}{a_{11}}$

Then $\cos(\theta) = a_{11}$ and $\sin(\theta) = a_{21}$.

(15) $R = R_{y,\pi/3} R_{x,\pi/2} = \begin{pmatrix} c_{\pi/2} & 0 & s_{\pi/2} \\ 0 & 1 & 0 \\ -s_{\pi/2} & 0 & c_{\pi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\pi/2} & -s_{\pi/2} \\ 0 & s_{\pi/2} & c_{\pi/2} \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$



(16) $$R_{23} = R_{12}^{T} R_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

(21) $$R_{x,\theta} R_{y,\phi} R_{z,\pi} R_{y,-\phi} R_{x,-\theta}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} -c_{2\phi} & -2 c_\phi s_\phi s_\theta & c_\theta s_{2\phi} \\ -2 c_\phi s_\phi s_\theta & -c_\theta^2 - c_{2\phi} s_\theta^2 & -c_\phi^2 s_{2\theta} \\ c_\theta s_{2\phi} & -c_\phi^2 s_{2\theta} & c_\phi^2 c_\theta^2 - c_\theta^2 s_\phi^2 - s_\theta^2 \end{bmatrix}$$

(35) $$g = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \begin{matrix} 3 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(36) $$g_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad g_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_{12} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_{02} = g_{01} g_{12} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \checkmark$$

(37) $$g_{01} = \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \\ \hline 0 & 1 \end{array}\right]; \quad g_{02} = \left(\begin{array}{c|c} I & \begin{matrix} -1/2 \\ 3/2 \\ 11/10 \end{matrix} \\ \hline 0 & 1 \end{array}\right)$$

$$g_{03} = \begin{bmatrix} 0 & 1 & 0 & -1/2 \\ 1 & 0 & 0 & 3/2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{23} = g_{02}^{-1} g_{03} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 19/10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$