ME 380: Kinematics and Machine Dynamics
Fall 2021 | Midterm Examination #2

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Question:	1	2	3	Total
Points:	40	60	0	100
Bonus Points:	0	0	25	25
Score:				

Angle sum and difference identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$

Let $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top \in \mathbb{R}^3$ be a vector. There is a unique correspondence between a 3×3 skew-skymmetric matrix, $S(a)$ and the vector a:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a \leftrightarrow S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Angular and linear velocities

We define the (body) angular velocity and angular acceleration of a reference frame Σ_B with respect to Σ_A, respectively, by

$$S\left({}^A\omega^B\right) = R_{ab}^\top \dot{R}_{ab}, \qquad {}^A\alpha^B = \frac{{}^A\mathrm{d}\,{}^A\omega^B}{\mathrm{d}t}.$$

Suppose that $r \in \mathbb{R}^3$ is a vector that begins at *any* point fixed in frame Σ_A and ends at a point, Q, moving in Σ_A. We define the linear velocity and acceleration of point Q in Σ_A, respectively, by

$${}^Av^Q \triangleq \frac{{}^A\mathrm{d}r}{\mathrm{d}t}, \qquad {}^Aa^Q \triangleq \frac{{}^A\mathrm{d}\,{}^Av^Q}{\mathrm{d}t}.$$

Differentiation in two frames

If Σ_A and Σ_B are two reference frames, the first time-derivative of *any* vector v in Σ_A and in Σ_B are related to each other as follows:

$$\frac{{}^A\mathrm{d}v}{\mathrm{d}t} = \frac{{}^B\mathrm{d}v}{\mathrm{d}t} + {}^A\omega^B \times v.$$

Auxiliary reference frames

The angular velocity of a rigid body B in a reference frame A can be expressed in the following form involving n auxiliary reference frames $A_1, \ldots, A_n$:

$${}^A\omega^B = {}^A\omega^{A_1} + {}^{A_1}\omega^{A_2} + \cdots + {}^{A_{n-1}}\omega^{A_n} + {}^{A_n}\omega^B.$$

Question 1 (16 minutes) .... *40 (main) + 0 (bonus) points*

Consider the window operator mechanism shown in Figure 1. Let ℓ_k denote the (constant) length of link $k \in \{1, 2, 3\}$, and let ℓ_4 and ℓ_4' denote the (constant) lengths of the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_2C}$, respectively. The distances a_{0x} and a_{0y} are also constants.

$$S({}^0\omega^4) = R_{04}^T \dot{R}_{04}$$

$$= \dot\theta_4 \begin{bmatrix} -s_4 & -c_4 & 0 \\ c_4 & -s_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_4 & -s_4 & 0 \\ s_4 & -c_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \dot\theta_4 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore {}^0\omega^4 = -\dot\theta_4 z_4$$

$R_{01} = R_{z,\theta_1}$

$R_{02} = R_{z,\theta_1+\theta_2}$

$R_{03} = R_{z,\theta_3}$

$$R_{04} = R_{z,-(\theta_4+\pi/2)} = \begin{bmatrix} -s_4 & c_4 & 0 \\ -c_4 & -s_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

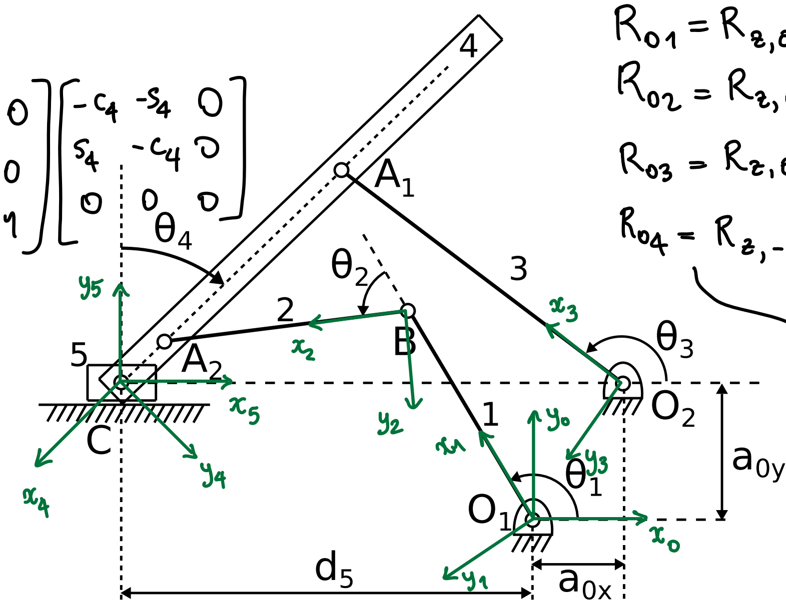


Figure 1: `Window operator linkage`

(a) [10 points] Analytically derive the loop equations in terms of the variables $\{\theta_i\}_{i=1}^4$, d_5, and the constant lengths.

(b) [30 points] Given ${}^0\omega^3$, write down the velocity-level loop equations, *in matrix form*, used to find ${}^0\omega^1$, ${}^0\omega^2$, ${}^0\omega^4$, and ${}^0v^C$.

(a) Loop 1 : $O_1 - B - A_2 - C - O_1$

$$\ell_1 x_1 + \ell_2 x_2 + \ell_4' x_4 + d_5 x_0 - a_{0y} y_0 = 0.$$

In Σ_0 :
$$\ell_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + \ell_2 \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix} - \ell_4' \begin{bmatrix} s_4 \\ c_4 \end{bmatrix} + \begin{bmatrix} d_5 \\ -a_{0y} \end{bmatrix} = 0. \qquad (*)$$

Loop 2: $O_2 - A_1 - C - O_2 \rightarrow \ell_3 x_3 + (\ell_4 + \ell_4') x_4 + (d_5 + a_{0x}) x_0 = 0$

In Σ_0:
$$\ell_3 \begin{bmatrix} c_3 \\ s_3 \end{bmatrix} - (\ell_4 + \ell_4') \begin{bmatrix} s_4 \\ c_4 \end{bmatrix} + \begin{bmatrix} d_5 + a_{0x} \\ 0 \end{bmatrix} = 0. \qquad (**)$$

(b) Differentiating Loop 1, we obtain

$$\underbrace{\dot{\theta}_1 z_1 \times \ell_1 x_1}_{\ell_1 \dot{\theta}_1 y_1} + \underbrace{(\dot{\theta}_1 + \dot{\theta}_2) z_2 \times \ell_2 x_2}_{\ell_2 (\dot{\theta}_1 + \dot{\theta}_2) y_2} + \underbrace{(-\dot{\theta}_4 z_4) \times \ell_4' x_4}_{-\ell_4' \dot{\theta}_4 y_4} + \dot{d}_5 x_0 = 0.$$

In Σ_0:

$$\ell_1 \dot{\theta}_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} + \ell_2 (\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -s_{12} \\ c_{12} \end{bmatrix} - \ell_4' \dot{\theta}_4 \begin{bmatrix} c_4 \\ -s_4 \end{bmatrix} + \begin{bmatrix} \dot{d}_5 \\ 0 \end{bmatrix} = 0$$

Differentiating Loop 2, we obtain

$$\underbrace{\dot{\theta}_3 z_3 \times \ell_3 x_3}_{\ell_3 \dot{\theta}_3 y_3} + \underbrace{(-\dot{\theta}_4 z_4) \times (\ell_4 + \ell_4') x_4}_{-(\ell_4 + \ell_4') \dot{\theta}_4 y_4} + \dot{d}_5 x_0 = 0.$$

In Σ_0:

$$\ell_3 \dot{\theta}_3 \begin{bmatrix} -s_3 \\ c_3 \end{bmatrix} - (\ell_4 + \ell_4') \dot{\theta}_4 \begin{bmatrix} c_4 \\ -s_4 \end{bmatrix} + \begin{bmatrix} \dot{d}_5 \\ 0 \end{bmatrix} = 0.$$

Putting these in matrix-vector form, we obtain

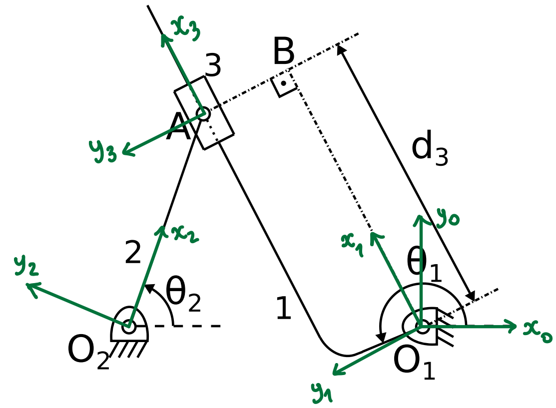
$$\begin{bmatrix} -\ell_1 s_1 - \ell_2 s_{12} & -\ell_2 s_{12} & -\ell_4' c_4 & 1 \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} & \ell_4' s_4 & 0 \\ 0 & 0 & -(\ell_4 + \ell_4') c_4 & 1 \\ 0 & 0 & (\ell_4 + \ell_4') s_4 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_4 \\ \dot{d}_5 \end{bmatrix} = \ell_3 \dot{\theta}_3 \begin{bmatrix} 0 \\ 0 \\ s_3 \\ -c_3 \end{bmatrix}.$$

Question 2 . . (18 minutes) . . . . *60 (main) + 0 (bonus) points*

Consider the sliding contact linkage, consisting of 3 rigid bodies, shown in the figure below. The constants pertinent to this problem are listed beside this figure.

Magnitude	Vector	Value [units]
l_0	$\left\lvert\overrightarrow{O_1O_2}\right\rvert$	1.5
l_1	$\left\lvert\overrightarrow{AB}\right\rvert$	0.5
l_2	$\left\lvert\overrightarrow{O_2A}\right\rvert$	1.25

Hint. The vector from O_1 to A can be decomposed into the sum of the vectors from O_1 to B and from B to A.



(a) [15 points] Derive the position-level loop equations for this mechanism in terms of the variables θ_1, θ_2, d_3, and the various constants l_0, l_1, and l_2. Now, if it is observed that $\theta_1 = -160°$ and $\theta_2 = 63.311830°$ then what is the value of d_3?

(b) [30 points] At this instant, if ${}^0\omega^1 = 1z_1$ rad/s, compute ${}^0\omega^2 = \dot{\theta}_2 z_2$ and ${}^1v^A = \dot{d}_3 x_1$. Here, z_1 and z_2 are the unit vectors, fixed on links 1 and 2, resp., that point out of the page; x_1 points from O_1 to B and point A is considered fixed on link 3 (or 2).

(c) [15 points] Derive the acceleration-level loop equations and use them to solve for ${}^0\alpha^2$ and ${}^1a^A$ assuming ${}^0\alpha^1 = -5z_1$ rad/s².

(a) $$d_3 x_1 + l_1 y_1 - l_2 x_2 + l_0 x_0 = 0 \Rightarrow d_3\begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} + l_1\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} - l_2\begin{bmatrix} c_2 \\ s_2 \end{bmatrix} + l_0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \quad (*)$$

Massaging this equation, we obtain

$$d_3\begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} = l_2\begin{bmatrix} c_2 \\ s_2 \end{bmatrix} - l_0\begin{bmatrix} 1 \\ 0 \end{bmatrix} - l_1\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \overset{\text{substitute numerics}}{=} \begin{bmatrix} -0.468736 \\ 1.2878402 \end{bmatrix}$$

Taking magnitudes on both sides, we obtain

$$d_3 = 1.370491$$

(b) Differentiating equation $(*)$, we have

$$\dot{d}_3\begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} + l_2\dot{\theta}_2\begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} = \dot{\theta}_1\left(-d_3\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + l_1\begin{bmatrix} s_1 \\ -c_1 \end{bmatrix}\right).$$

Then we have $J\mathbf{x} = b$, where (**)

$$J = \begin{bmatrix} s_1 & l_2 s_2 \\ -c_1 & -l_2 c_2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \dot{d}_3 \\ \dot{\theta}_2 \end{bmatrix}, \quad b = \dot{\theta}_1 \left(l_1 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} - d_3 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \right)$$

Substituting numerics and solving for $\mathbf{x}$, we obtain

$$\mathbf{x} = \begin{bmatrix} 1.953728 \\ 1.598313 \end{bmatrix}$$

12 min.

(c) Differentiating (**), we obtain

$$J\dot{\mathbf{x}} + \dot{J}\mathbf{x} = \dot{b} \quad \text{or} \quad J\dot{\mathbf{x}} = \dot{b} - \dot{J}\mathbf{x}.$$

Now, $\dot{J} = \begin{bmatrix} c_1 \dot{\theta}_1 & l_2 c_2 \dot{\theta}_2 \\ s_1 \dot{\theta}_1 & l_2 s_2 \dot{\theta}_2 \end{bmatrix}$ and

$$\dot{b} = \ddot{\theta}_1 \left(l_1 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} - d_3 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \right) + \dot{\theta}_1 \left(l_1 \dot{\theta}_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} - \dot{d}_3 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} - d_3 \dot{\theta}_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} \right)$$

Substituting the numerics, we obtain $\dot{b} - \dot{J}\mathbf{x} = \begin{bmatrix} -4.686829 \\ -2.907865 \end{bmatrix}$

so that

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddot{d}_3 \\ \ddot{\theta}_2 \end{bmatrix} = J^{-1}(\dot{b} - \dot{J}\mathbf{x}) = \begin{bmatrix} -6.856129 \\ -6.296179 \end{bmatrix}.$$

18 min.

Question 3 (13 minutes) .... 25 (bonus) points

Figure 2 shows a schematic of the Oldham coupling, where r_0, r_1, and r_2 denote the lengths of the base, first, and the second links, respectively. The angle $\phi = \pi/4$ and the length $r_0 = 3/2$ in are fixed. If $\theta_1 = 5\pi/12$, $\omega_1 = \dot{\theta}_1 = -3/5$ rad/s, and $\alpha_1 = \ddot{\theta}_1 = 2$ rad/s², then perform the following analysis.

(a) [5 points (bonus)] Write down the loop equations and solve for r_1 and r_2 at this instant.

(b) [10 points (bonus)] Find $\dot{r}_1$ and $\dot{r}_2$.

(c) [10 points (bonus)] Find $\ddot{r}_1$ and $\ddot{r}_2$.

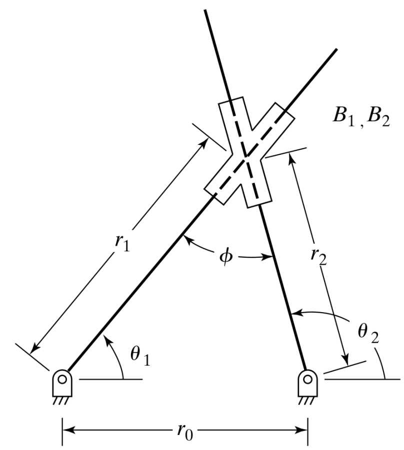


Figure 2: Oldham coupling

(a)
$$r_1 c_1 - r_2 c_2 - r_0 = 0$$
$$r_1 s_1 - r_2 s_2 = 0$$
where $\theta_2 = \theta_1 + \phi$ or (*)

$$\begin{bmatrix} c_1 & -c_2 \\ s_1 & -s_2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} r_0 \\ 0 \end{bmatrix} \quad \text{or} \quad J\mathbf{X} = b, \quad \mathbf{X} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Substituting numerics, we get

$$\boxed{\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1.837117 \\ 2.049038 \end{bmatrix}}$$

(b) Differentiating (*), we obtain [Note: $\dot{\theta}_2 = \dot{\theta}_1$]

$$\begin{bmatrix} c_1 & -c_2 \\ s_1 & -s_2 \end{bmatrix}\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} + \underbrace{\dot{\theta}_1 \begin{bmatrix} -s_1 & s_2 \\ c_1 & -c_2 \end{bmatrix}}_{\dot{J}}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$J\dot{X} = \dot{\theta}_1 \begin{bmatrix} 0 \\ -r_0 \end{bmatrix} . \qquad (**)$$

Substituting the numerics,

$$\boxed{\dot{X} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} 0.636396 \\ -0.329423 \end{bmatrix}}$$

(c) Differentiating (**), we obtain

$$J\ddot{X} + \dot{J}\dot{X} = \ddot{\theta}_1 \begin{bmatrix} 0 \\ -r_0 \end{bmatrix}$$

$$\dot{J}\dot{x} = \dot{\theta}_1 \begin{bmatrix} -s_1 & s_2 \\ c_1 & -c_2 \end{bmatrix} \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \overset{\text{from } (**)}{=} \begin{bmatrix} r_0 \dot{\theta}_1^2 \\ 0 \end{bmatrix}$$

so, we get

$$J\ddot{X} = -r_0 \begin{bmatrix} \dot{\theta}_1^2 \\ \ddot{\theta}_1 \end{bmatrix}$$

Substituting numerics, we obtain

$$\boxed{\ddot{X} = \begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} = \begin{bmatrix} -2.782683 \\ 0.360422 \end{bmatrix} .}$$