ME 380: Kinematics and Machine Dynamics
Fall 2020 / Midterm Examination #2

Name: ______________________________

Question:	1	2	3	4	Total
Points:	15	40	45	0	100
Bonus Points:	5	0	0	15	20

Angle sum and difference identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$

Let $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top \in \mathbb{R}^3$ be a vector. There is a unique correspondence between a 3×3 skew-skymmetric matrix, $\widehat{a}$ and the vector a:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a \leftrightarrow \widehat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Angular and linear velocities

We define the (body) angular velocity and angular acceleration of a reference frame Σ_B with respect to Σ_A, respectively, by

$${}^A\widehat{\omega}^B = R_{ab}^\top \dot{R}_{ab}, \qquad {}^A\alpha^B = \frac{{}^A\mathrm{d}\,{}^A\omega^B}{\mathrm{d}t}.$$

Suppose that $r \in \mathbb{R}^3$ is a vector that begins at *any* point fixed in frame Σ_A and ends at a point, Q, moving in Σ_A. We define the linear velocity and acceleration of point Q in Σ_A, respectively, by

$${}^A v^Q \triangleq \frac{{}^A\mathrm{d}r}{\mathrm{d}t}, \qquad {}^A a^Q \triangleq \frac{{}^A\mathrm{d}\,{}^A v^Q}{\mathrm{d}t}.$$

Differentiation in two frames

If Σ_A and Σ_B are two reference frames, the first time-derivative of *any* vector v in Σ_A and in Σ_B are related to each other as follows:

$$\frac{{}^A\mathrm{d}v}{\mathrm{d}t} = \frac{{}^B\mathrm{d}v}{\mathrm{d}t} + {}^A\omega^B \times v.$$

Auxiliary reference frames

The angular velocity of a rigid body B in a reference frame A can be expressed in the following form involving n auxiliary reference frames $A_1, \ldots, A_n$:

$${}^A\omega^B = {}^A\omega^{A_1} + {}^{A_1}\omega^{A_2} + \cdots + {}^{A_{n-1}}\omega^{A_n} + {}^{A_n}\omega^B.$$

Question 1 .... *15 (main) + 5 (bonus) points*

Figure 1 shows a schematic of the Oldham coupling, where r_0, r_1, and r_2 denote the lengths of the base, first, and the second links, respectively. The angle $\phi = \frac{\pi}{4}$ and the length $r_0 = \frac{3}{2}$ [in] are fixed. If $\theta_1 = \frac{5\pi}{12}$, $\omega_1 = \dot{\theta}_1 = -\frac{3}{5}\ \frac{\text{rad}}{\text{s}}$, and $\alpha_1 = \ddot{\theta}_1 = 2\ \frac{\text{rad}}{\text{s}^2}$, then perform the following analysis.

(a) [7 points] Write down the loop equations and solve for r_1 and r_2 at this instant.

(b) [8 points] Find $\dot{r}_1$ and $\dot{r}_2$.

(c) [5 points (bonus)] Find $\ddot{r}_1$ and $\ddot{r}_2$.

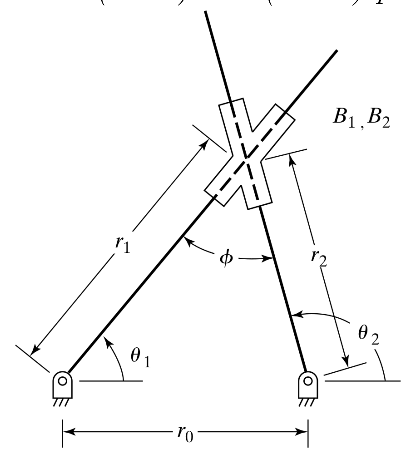


Figure 1: Oldham coupling

Solution: (a) Notice that $\theta_2 = \theta_1 + \phi = \frac{2\pi}{3}$. The loop equations are

$$\begin{aligned} r_1 \cos\theta_1 - r_2 \cos\theta_2 &= r_0, \\ r_1 \sin\theta_1 - r_2 \sin\theta_2 &= 0. \end{aligned} \tag{1}$$

Plugging in the values and solving yields $\boxed{r_1 = 1.837 \text{ in}}$ and $\boxed{r_2 = 2.049 \text{ in}}$.

(b) Differentiating the position-level loop equations (1) w.r.t. time yields the velocity-level equations

$$\begin{aligned} \dot{r}_1 \cos\theta_1 - \dot{r}_2 \cos\theta_2 &= 0, \\ \dot{r}_1 \sin\theta_1 - \dot{r}_2 \sin\theta_2 &= -r_0\dot{\theta}_1, \end{aligned} \tag{2}$$

where we used the position-level loop equations (1) to simplify the right-hand side expressions. Plugging in the values and solving yields $\boxed{\dot{r}_1 = 0.636\ \frac{\text{in}}{\text{s}}}$ and $\boxed{\dot{r}_2 = -0.329\ \frac{\text{in}}{\text{s}}}$.

(c) Differentiating the velocity-level loop equations (2) w.r.t. time yields the acceleration level equations

$$\begin{aligned} \ddot{r}_1 \cos\theta_1 - \ddot{r}_2 \cos\theta_2 &= -r_0\dot{\theta}_1^2, \\ \ddot{r}_1 \sin\theta_1 - \ddot{r}_2 \sin\theta_2 &= -r_0\ddot{\theta}_1, \end{aligned} \tag{3}$$

where we used the position- and velocity-level loop equations (1)-(2) to simplify the right-hand side expressions. Plugging in the values and solving yields $\boxed{\ddot{r}_1 = -2.783\ \frac{\text{in}}{\text{s}^2}}$ and $\boxed{\ddot{r}_2 = 0.360\ \frac{\text{in}}{\text{s}^2}}$.

Question 2 .... *40 points*

Consider the mechanism shown in Figure 2. Excluding the fixed link, it consists of 5 links, one of which is a slider. Let l_k denote the length of link $k \in \{1, 2, 3\}$, and let l_4 and l_4' denote the lengths of the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_2C}$, respectively.

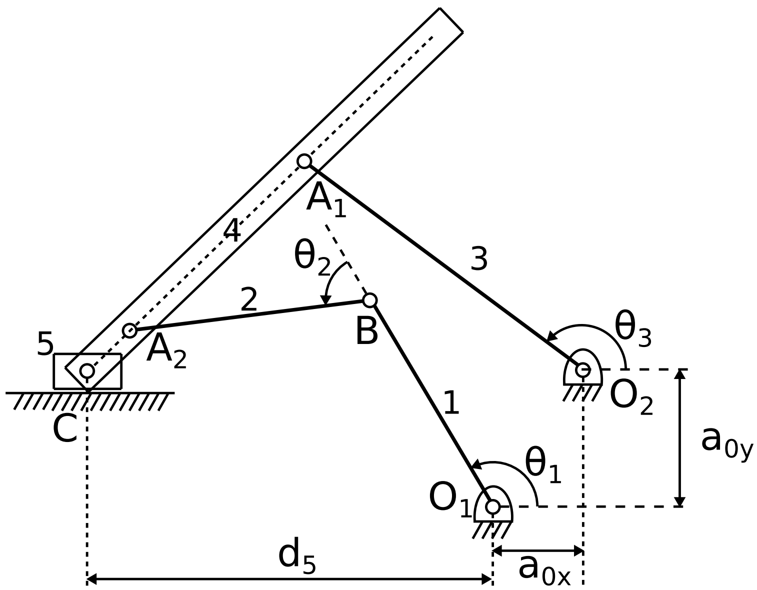


Figure 2: `Window operator linkage`

(a) [5 points] If link 3 is the input and link 5 is the output, state which link is the coupler. Also, clearly mark the transmission angle by the quantity φ.

> **Solution:** The transmission angle φ is the angle between the vertical axis and the unit vector along $\overrightarrow{CA_1}$. The coupler is link 4.

(b) [15 points] Analytically derive the loop equations in terms of the variables $\theta_1, \theta_2, \theta_3$, the transmission angle φ, d_5, and the various constants $l_1, \ldots, l_4$, l_4', a_{0x}, and a_{0y}.

> **Solution:**
> Loop 1:
> $$\frac{O_2 - A_1 - C - O_2}{r^{O_2A_1} + r^{A_1C} + r^{CO_2} = 0} \qquad l_3 \begin{bmatrix} c_3 \\ s_3 \end{bmatrix} - (l_4 + l_4') \begin{bmatrix} s_\varphi \\ c_\varphi \end{bmatrix} + \begin{bmatrix} d_5 + a_{0x} \\ 0 \end{bmatrix} = 0, \tag{4}$$
> Loop 2:
> $$\frac{O_1 - B - A_2 - C - O_1}{r^{O_1B} + r^{BA_2} + r^{A_2C} + r^{CO_1} = 0} \qquad l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + l_2 \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix} - l_4' \begin{bmatrix} s_\varphi \\ c_\varphi \end{bmatrix} + \begin{bmatrix} d_5 \\ -a_{0y} \end{bmatrix} = 0. \tag{5}$$

(c) [20 points] Given ${}^0\omega^3$, write down the velocity-level loop equations, *in matrix form*, used to find ${}^0\omega^1$, ${}^0\omega^2$, ${}^0\omega^4$, and ${}^0v^C$.

Solution: Note that ${}^0\omega^4 = -\dot{\varphi} z_4$. The time derivative of Loop 1 w.r.t. the 0^{th} frame is

$$
{}^0\omega^3 \times r^{O_2A_1} + {}^0\omega^4 \times r^{A_1C} + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{CO_2} = 0.
$$

Expressing in the 0^{th} frame gives

$$
\begin{aligned}
-l_3 s_3 \dot{\theta}_3 - (l_4 + l_4') c_\varphi \dot{\varphi} + \dot{d}_5 &= 0, \\
l_3 c_3 \dot{\theta}_3 + (l_4 + l_4') s_\varphi \dot{\varphi} &= 0.
\end{aligned}
$$

The time derivative of Loop 2 w.r.t. the 0^{th} frame is

$$
{}^0\omega^1 \times r^{O_1B} + {}^0\omega^2 \times r^{BA_2} + {}^0\omega^4 \times r^{A_2C} + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{CO_1} = 0.
$$

Expressing in the 0^{th} frame gives

$$
\begin{aligned}
-l_1 s_1 \dot{\theta}_1 - l_2 s_{12} (\dot{\theta}_1 + \dot{\theta}_2) - l_4' c_\varphi \dot{\varphi} + \dot{d}_5 &= 0, \\
l_1 c_1 \dot{\theta}_1 + l_2 c_{12} (\dot{\theta}_1 + \dot{\theta}_2) + l_4' s_\varphi \dot{\varphi} &= 0.
\end{aligned}
$$

In matrix form,

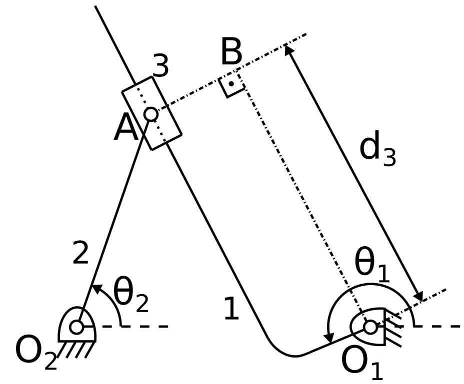
$$
\begin{bmatrix}
0 & 0 & -(l_4 + l_4') c_\varphi & 1 \\
0 & 0 & (l_4 + l_4') s_\varphi & 0 \\
-l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & -l_4' c_\varphi & 1 \\
l_1 c_1 + l_2 c_{12} & l_2 c_{12} & l_4' s_\varphi & 0
\end{bmatrix}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\varphi} \\ \dot{d}_5 \end{bmatrix}
= l_3 \dot{\theta}_3 \begin{bmatrix} s_3 \\ -c_3 \\ 0 \\ 0 \end{bmatrix}.
$$

Question 3 . . . . *45 points*

Consider the sliding contact linkage, consisting of 3 rigid bodies, shown in the figure below. The constants pertinent to this problem are listed beside this figure.

Quantity	Endpoints	Length [units]
l_0	$\lvert O_1O_2 \rvert$	1.5
l_1	$\lvert AB \rvert$	0.5
l_2	$\lvert O_2A \rvert$	1.25

Hint. The vector from O_1 to A can be decomposed into the sum of the vectors from O_1 to B and from B to A.



(a) [15 points] Analytically derive the position-level loop equations for this mechanism in terms of the variables θ_1, θ_2, d_3, and the various constants l_0, l_1, and l_2.

Solution: Let $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AO_2}$, and $p_0 = \overrightarrow{O_2O_1}$. We have

$$p_0 + p_1 + r^{A_1A_2} + p_2 = 0. \tag{6}$$

Writing these equations in coordinates gives

$$l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + d_3 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} - l_2 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} + \begin{bmatrix} l_0 \\ 0 \end{bmatrix} = 0.$$

(b) [15 points] Consider an instant when $\theta_1 = -160°$. The position-level kinematics is solved with the solution $\theta_2 = 63.312°$ and $d_3 = 1.370$. At this instant, if ${}^0\omega^1 = 10z_1 \frac{\text{rad}}{\text{s}}$, compute ${}^0\omega^2 = \dot{\theta}_2 z_2$ and ${}^1v^A = \dot{d}_3 x_1$, where z_1 and z_2 are unit vectors, fixed on links 1 and 2, respectively, that point out of the page, x_1 points from O_1 to B and point A is considered fixed on link 3 (or 2).

Solution: Differentiating w.r.t. time equations (6) in frame 0 gives

$${}^0\omega^1 \times p_1 + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{A_1A_2} + {}^0\omega^2 \times p_2 = 0.$$

This is equivalent to

$${}^0\omega^1 \times p_1 + {}^1v^{A_2} + {}^0\omega^1 \times r^{A_1A_2} + {}^0\omega^2 \times p_2 = 0. \tag{7}$$

Writing these equations in coordinates gives

$$\dot{\theta}_1 \begin{bmatrix} -(l_1 s_1 - d_3 c_1) \\ l_1 c_1 + d_3 s_1 \end{bmatrix} + \dot{d}_3 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} + l_2 \dot{\theta}_2 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} = 0.$$

In matrix form, we have

$$\begin{bmatrix} s_1 & l_2 s_2 \\ -c_1 & -l_2 c_2 \end{bmatrix} \begin{bmatrix} \dot{d}_3 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} l_1 s_1 - d_3 c_1 \\ -(l_1 c_1 + d_3 s_1) \end{bmatrix} \dot{\theta}_1.$$

Plugging in the numbers, the answer turns out to be $\dot{\theta}_2 = 15.983\ \frac{\text{rad}}{\text{s}}$, $\dot{d}_3 = 19.537$ [units].

(c) [15 points] Write down the acceleration-level loop equations in *vector form* (You do not need to express them in coordinates).

Solution: Differentiation w.r.t. time equations (7) in frame 0 gives

$$\begin{aligned} &{}^0\alpha^1 \times p_1 + {}^0\omega^1 \times \left({}^0\omega^1 \times p_1\right) + {}^1a^{A_2} + 2\,{}^0\omega^1 \\ &\times {}^1v^{A_2} + {}^0\alpha^2 \times p_2 + {}^0\omega^2 \times \left({}^0\omega^2 \times p_2\right) = 0. \end{aligned}$$

Question 4 . . . . *15 (bonus) points*

Consider a circular hoop of radius R, rotating about the world z-axis as shown in Figure 3. A bead traverses the circular hoop along a groove. Here are some facts to help you with the question:

- φ is the angle between x-axes of the world frame and the hoop frame.
- When $\varphi = 0$, the orientation of the hoop is identical to the world.
- θ is the angle between the z-axes of the hoop frame and the bead frame such that the x-axis of the bead frame points from the origin of the world frame to the center of the bead.
- When $\theta = 0$, the orientation of the bead is identical to the hoop.
- When $\varphi = \theta = 0$, the translation of the bead, in the world frame is given by $p_{bead} = \begin{bmatrix} R & 0 & 0 \end{bmatrix}^\top$.

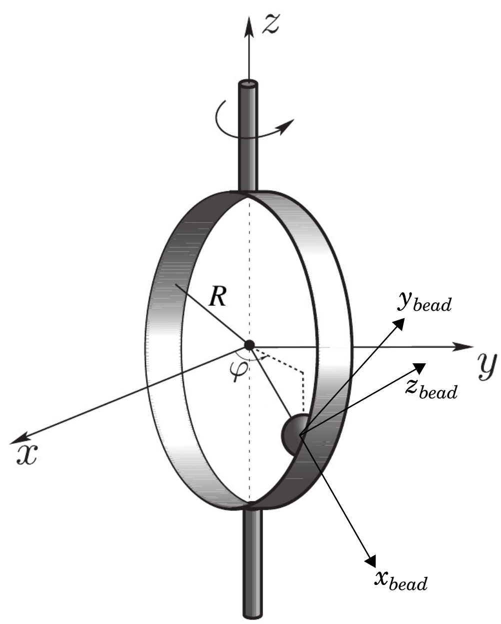


Figure 3: `Bead on a rotating hoop`

(a) [10 points (bonus)] Find the (x, y, z) position of the bead in the world frame as a function of φ and θ. Also, find the velocity $(\dot{x}, \dot{y}, \dot{z})$ of the bead in the world frame as a function of φ, θ, $\dot{\varphi}$, and $\dot{\theta}$.

> **Solution:** Let p be the position vector from the origin to the bead. In the bead frame, this vector is given by $p = R \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$. We express this vector in the world frame by
>
> $$p^{world} = R^{world}_{bead} p.$$

By the rule of composition of rotations, we have $R_{bead}^{world} = R_{hoop}^{world} R_{bead}^{hoop} = R_{z,\varphi} R_{y,\theta}$. Computing this last expression, we get

$$R_{bead}^{world} = \begin{bmatrix} c_\varphi c_\theta & -s_\varphi & c_\varphi s_\theta \\ s_\varphi c_\theta & c_\varphi & s_\varphi s_\theta \\ -s_\theta & 0 & c_\theta \end{bmatrix}.$$

Therefore, we have

$$p^{world} = R \begin{bmatrix} c_\varphi c_\theta \\ s_\varphi c_\theta \\ -s_\theta \end{bmatrix} \tag{8}$$

Differentiating this vector in the world frame gives the velocity of the bead in the world frame

$$^{world}v^{bead} = {}^{world}\frac{\mathrm{d}}{\mathrm{d}t} p^{world} = R \begin{bmatrix} -s_\varphi c_\theta & -c_\varphi s_\theta \\ c_\varphi c_\theta & -s_\varphi s_\theta \\ 0 & -c_\theta \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \end{bmatrix}.$$

(b) [5 points (bonus)] Suppose that $\varphi(0) = \frac{2\pi}{5}$ and the circular hoop has a constant angular speed with respect to the world, i.e., $\omega = \dot\varphi = \frac{\pi}{2} \ \frac{\text{rad}}{\text{s}}$. Suppose further that the bead has a constant angular speed with respect to the hoop, i.e., $^{hoop}\omega^{bead} = \dot\theta y_{hoop}$ such that $\theta(0) = \frac{\pi}{6}$ and $\theta(3) = -\frac{2\pi}{3}$. Use this information to express φ and θ as a function of time.

Solution:

$$\varphi(t) \ = \varphi(0) + \dot\varphi t = \frac{2\pi}{5} + \frac{\pi}{2} t.$$

Next, we have

$$\theta(t) \ = \theta(0) + \dot\theta t \Longrightarrow \theta(3) = \theta(0) + 3\dot\theta.$$

From this, we find that $\dot\theta = -\frac{5\pi}{18}$. Plugging this in the expression for $\theta(t)$, we get

$$\theta(t) = \frac{\pi}{6} - \frac{5\pi}{18} t.$$