ME 380
Kinematics and Machine Dynamics
Midterm II Solutions

1. [35 points] Consider the mechanism shown in Figure 1. Excluding the fixed link, it consists of 5 links, one of which is a slider. Let l_k denote the length of link $k \in \{1, 2, 3\}$, and let l_4 and l_4' denote the lengths of the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_2C}$, respectively.

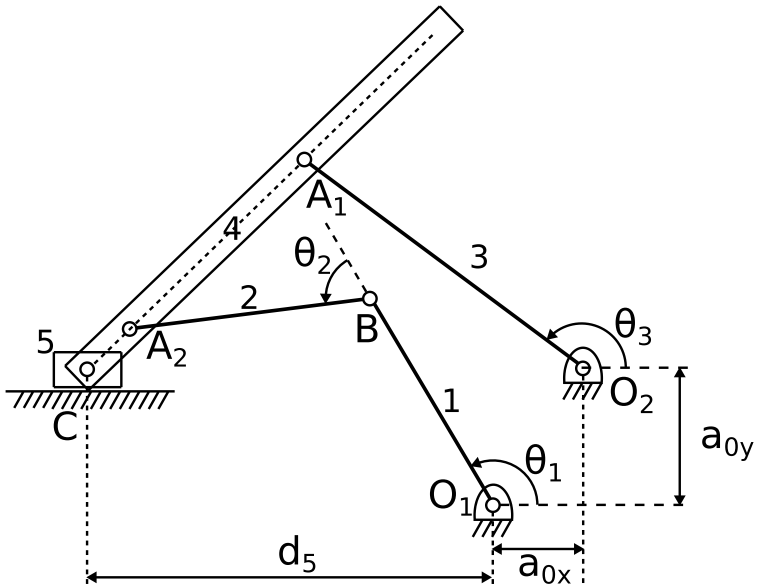


Figure 1: `Window operator linkage`

(a) [5 points] If link 3 is the input and link 5 is the output, state which link is the coupler. Also, clearly mark the transmission angle by the quantity φ.

Solution: The transmission angle φ is the angle between the vertical axis and the unit vector along $\overrightarrow{CA_1}$.

(b) [15 points] Analytically derive the loop equations in terms of the variables $\theta_1, \theta_2, \theta_3$, the transmission angle φ, d_5, and the various constants $l_1, \ldots, l_4$, l_4', a_{0x}, and a_{0y}.

Solution:
Loop 1:

$\frac{O_2 - A_1 - C - O_2}{r^{O_2A_1} + r^{A_1C} + r^{CO_2} = 0}$

$$l_3 \begin{bmatrix} c_3 \\ s_3 \end{bmatrix} - (l_4 + l_4') \begin{bmatrix} s_\varphi \\ c_\varphi \end{bmatrix} + \begin{bmatrix} d_5 + a_{0x} \\ 0 \end{bmatrix} = 0, \quad (1)$$

Loop 2:

$\frac{O_1 - B - A_2 - C - O_1}{r^{O_1B} + r^{BA_2} + r^{A_2C} + r^{CO_1} = 0}$

$$l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + l_2 \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix} - l_4' \begin{bmatrix} s_\varphi \\ c_\varphi \end{bmatrix} + \begin{bmatrix} d_5 \\ -a_{0y} \end{bmatrix} = 0. \quad (2)$$

(c) [15 points] Given ${}^0\omega^3$, write down the velocity-level loop equations, *in matrix form*, used to find ${}^0\omega^1$, ${}^0\omega^2$, $\dot{\varphi}$, and $\dot{d}_5$,.

Solution: Note that ${}^0\omega^4 = -\dot{\varphi} z_4$. The time derivative of Loop 1 w.r.t. the 0$^{\text{th}}$ frame is

$$ {}^0\omega^3 \times r^{O_2A_1} + {}^0\omega^4 \times r^{A_1C} + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{CO_2} = 0. $$

Expressing in the 0$^{\text{th}}$ frame gives

$$ \begin{aligned} -l_3 s_3 \dot{\theta}_3 - (l_4 + l_4')c_\varphi \dot{\varphi} + \dot{d}_5 &= 0, \\ l_3 c_3 \dot{\theta}_3 + (l_4 + l_4')s_\varphi \dot{\varphi} &= 0. \end{aligned} $$

The time derivative of Loop 2 w.r.t. the 0$^{\text{th}}$ frame is

$$ {}^0\omega^1 \times r^{O_1B} + {}^0\omega^2 \times r^{BA_2} + {}^0\omega^4 \times r^{A_2C} + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{CO_1} = 0. $$

Expressing in the 0$^{\text{th}}$ frame gives

$$ \begin{aligned} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) - l_4' c_\varphi \dot{\varphi} + \dot{d}_5 &= 0, \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) + l_4' s_\varphi \dot{\varphi} &= 0. \end{aligned} $$

In matrix form,

$$ \begin{bmatrix} 0 & 0 & -(l_4 + l_4')c_\varphi & 1 \\ 0 & 0 & (l_4 + l_4')s_\varphi & 0 \\ -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & -l_4' c_\varphi & 1 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & l_4' s_\varphi & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\varphi} \\ \dot{d}_5 \end{bmatrix} = l_3 \dot{\theta}_3 \begin{bmatrix} s_3 \\ -c_3 \\ 0 \\ 0 \end{bmatrix}. $$

(d) [5 points (bonus)] Set up an optimization problem, whose solution yields the minimum length, $l_{4_{min}}$, of $\overrightarrow{A_1A_2}$, such that the transmission angle satisfies $\frac{\pi}{4} \le \varphi \le \frac{3\pi}{4}$. (Hint: Does not require computation, just write down an objective function and some necessary constraints.)

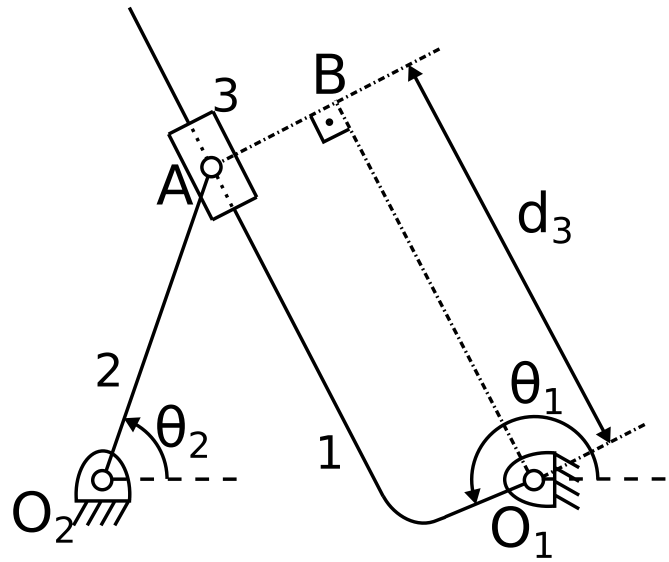
Solution:

$$ \text{minimize} \quad l_4 $$

$$
\begin{aligned}
\text{subject to} \quad & (1) \text{ and } (2), \\
& \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \\
& 0 \leq l_4.
\end{aligned}
$$

2. [40 points] Consider the sliding contact linkage, consisting of 3 rigid bodies, shown in the figure below. The constants pertinent to this problem are listed beside this figure.

Quantity	Endpoints	Length [units]
l_0	$\lvert O_1O_2 \rvert$	1.5
l_1	$\lvert AB \rvert$	0.5
l_2	$\lvert O_2A \rvert$	1.25



(a) [15 points] Analytically derive the position-level loop equations for this mechanism in terms of the variables θ_1, θ_2, d_3, and the various constants l_0, l_1, and l_2.

Solution: Let $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AO_2}$, and $p_0 = \overrightarrow{O_2O_1}$. We have

$$p_0 + p_1 + r^{A_1A_2} + p_2 = 0. \tag{3}$$

Writing these equations in coordinates gives

$$l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + d_3 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} - l_2 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} + \begin{bmatrix} l_0 \\ 0 \end{bmatrix} = 0.$$

(b) [15 points] Consider an instant when $\theta_1 = -160°$. The position-level kinematics is solved with the solution $\theta_2 = 63.312°$ and $d_3 = 1.370$. At this instant, if ${}^0\omega^1 = 10 z_0 \, \frac{\text{rad}}{\text{s}}$ (z_0 is the axis in the world frame that points perpendicular to the plane of motion), compute ${}^0\omega^2$ and $\dot{d}_3$.

Solution: Differentiating w.r.t. time equations (3) in frame 0 gives

$$ {}^0\omega^1 \times p_1 + \frac{{}^0\mathrm{d}}{\mathrm{d}t} r^{A_1A_2} + {}^0\omega^2 \times p_2 = 0. $$

This is equivalent to

$$ {}^0\omega^1 \times p_1 + {}^1v^{A_2} + {}^0\omega^1 \times r^{A_1A_2} + {}^0\omega^2 \times p_2 = 0. \tag{4} $$

Writing these equations in coordinates gives

$$ \dot{\theta}_1 \begin{bmatrix} -(l_1 s_1 - d_3 c_1) \\ l_1 c_1 + d_3 s_1 \end{bmatrix} + \dot{d}_3 \begin{bmatrix} s_1 \\ -c_1 \end{bmatrix} + l_2 \dot{\theta}_2 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} = 0. $$

In matrix form, we have

$$ \begin{bmatrix} s_1 & l_2 s_2 \\ -c_1 & -l_2 c_2 \end{bmatrix} \begin{bmatrix} \dot{d}_3 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} l_1 s_1 - d_3 c_1 \\ -(l_1 c_1 + d_3 s_1) \end{bmatrix} \dot{\theta}_1. $$

Plugging in the numbers, the answer turns out to be $\dot{\theta}_2 = 15.983\ \frac{\mathrm{rad}}{\mathrm{s}}$, $\dot{d}_3 = 19.537$ [units].

(c) [10 points] Write down the acceleration-level loop equations in *vector form* (You do not need to express them in coordinates).

Solution: Differentiation w.r.t. time equations (4) in frame 0 gives

$$ \begin{aligned} &{}^0\alpha^1 \times p_1 + {}^0\omega^1 \times \left({}^0\omega^1 \times p_1\right) + {}^1a^{A_2} + 2{}^0\omega^1 \\ &\times {}^1v^{A_2} + {}^0\alpha^2 \times p_2 + {}^0\omega^2 \times \left({}^0\omega^2 \times p_2\right) = 0. \end{aligned} $$

3. [25 points] Consider a circular hoop of radius R, rotating about the world z-axis as shown in Figure 2. A bead traverses the circular hoop along a groove. Here are some facts to help you with the question:

- φ is the angle between x-axes of the world frame and the hoop frame.
- When $\varphi = 0$, the orientation of the hoop is identical to the world.
- θ is the angle between the z-axes of the hoop frame and the bead frame, which undergoes a simple (basic) rotation about the y-axis of the hoop frame.
- When $\theta = 0$, the orientation of the bead is identical to the hoop.

- When $\varphi = \theta = 0$, the translation of the bead, in the world frame is given by $p_{bead} = \begin{bmatrix} R & 0 & 0 \end{bmatrix}^\top$.

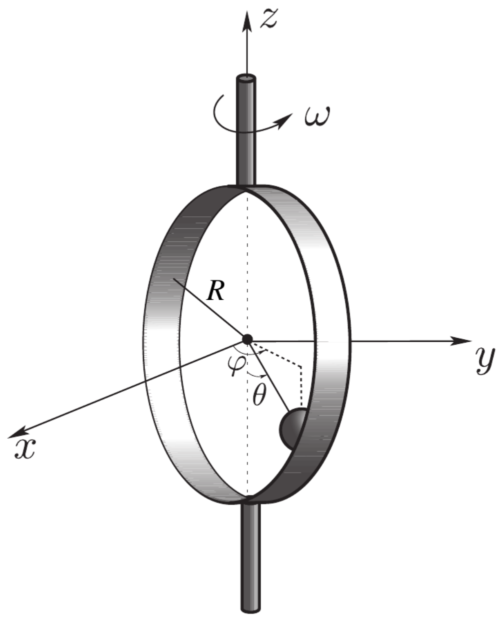


Figure 2: `Bead on a rotating hoop`

(a) [20 points] Find the (x, y, z) position of the bead in the world frame as a function of φ and θ. Also, find the velocity $(\dot{x}, \dot{y}, \dot{z})$ of the bead in the world frame as a function of φ, θ, $\dot{\varphi}$, and $\dot{\theta}$.

Solution: Let p be the position vector from the origin to the bead. In the bead frame, this vector is given by $p = R \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$. We express this vector in the world frame by

$$p^{world} = R^{world}_{bead}\, p.$$

By the rule of composition of rotations, we have $R^{world}_{bead} = R^{world}_{hoop} R^{hoop}_{bead} = R_{z,\varphi} R_{y,\theta}$. Computing this last expression, we get

$$R^{world}_{bead} = \begin{bmatrix} c_\varphi c_\theta & -s_\varphi & c_\varphi s_\theta \\ s_\varphi c_\theta & c_\varphi & s_\varphi s_\theta \\ -s_\theta & 0 & c_\theta \end{bmatrix}.$$

Therefore, we have

$$p^{world} = R \begin{bmatrix} c_\varphi c_\theta \\ s_\varphi c_\theta \\ -s_\theta \end{bmatrix} \tag{5}$$

Differentiating this vector in the world frame gives the velocity of the bead in the world frame

$$^{world}v^{bead} = \frac{^{world}\mathrm{d}}{\mathrm{d}t} p^{world} = R \begin{bmatrix} -s_\varphi c_\theta & -c_\varphi s_\theta \\ c_\varphi c_\theta & -s_\varphi s_\theta \\ 0 & -c_\theta \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \end{bmatrix}.$$

(b) [5 points] Suppose that $\varphi(0) = \frac{2\pi}{5}$ and the circular hoop has a constant angular speed with respect to the world, i.e., $\omega = \dot\varphi = \frac{\pi}{2}\ \frac{\text{rad}}{\text{s}}$. Suppose further that the bead has a constant angular speed with respect to the hoop, i.e., $^{hoop}\omega^{bead} = \dot\theta y_{hoop}$ such that $\theta(0) = \frac{\pi}{6}$ and $\theta(3) = -\frac{2\pi}{3}$. Use this information to express φ and θ as a function of time.

Solution:

$$\varphi(t) = \varphi(0) + \dot\varphi t = \frac{2\pi}{5} + \frac{\pi}{2} t.$$

Next, we have

$$\theta(t) = \theta(0) + \dot\theta t \Longrightarrow \theta(3) = \theta(0) + 3\dot\theta.$$

From this, we find that $\dot\theta = -\frac{5\pi}{18}$. Plugging this in the expression for $\theta(t)$, we get

$$\theta(t) = \frac{\pi}{6} - \frac{5\pi}{18} t.$$

(c) [5 points (bonus)] Making use of parts (a) and (b) compute the position and velocity of the bead in the world frame at $t = 2.5$ sec.

Solution: Plugging in $t = 2.5$ in the expressions derived in part (b), we get $\varphi = 5.183627878423159$ and $\theta = -1.6580627893946134$. Plugging in these values in the expressions derived in part (a), we get

$$p^{world}(2.5) = R \begin{bmatrix} -0.03956787920518083 \\ 0.07765633540864672 \\ 0.9961946980917455 \end{bmatrix}$$

$$\boxed{{}^{world}v^{bead} = R \begin{bmatrix} -0.5166561460473202 \\ 0.712437983853483 \\ -0.07605773364839047 \end{bmatrix}}$$