ME 380: Kinematics and Machine Dynamics
Fall 2021 / Midterm Examination #1

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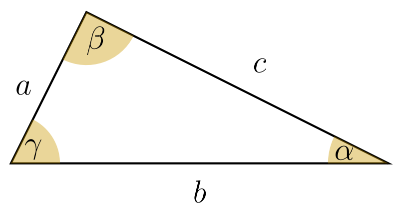
Angle sum and difference identities

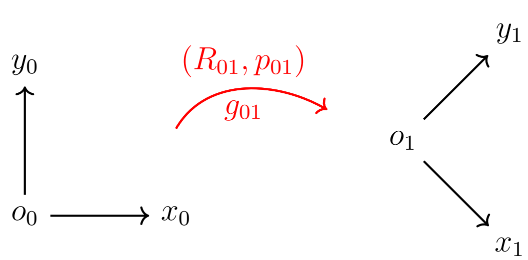
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Rigid Body Transformations



$\mathbb{R}^3 \ni p_{01}$: position vector from o_0 to o_1.

$$\mathbb{R}^{3\times 3} \ni R_{01} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$g_{01} = \left[\begin{array}{c|c} R_{01} & p_{01} \\ \hline 0 & 1 \end{array}\right]$$

Composition rules: $R_{ac} = R_{ab}R_{bc}$ and $g_{ac} = g_{ab}g_{bc}$.

Vector Operations

Let v and w be vectors in $\mathbb{R}^3$, then the inner (dot) product of v and w is defined by

$$v \cdot w = \|v\|\,\|w\|\cos\theta,$$

where θ is the angle between v and w.

Let Σ_a be a coordinate frame, and v be a vector in $\mathbb{R}^3$. Then, the expression of v in the coordinate frame Σ_a is given by

$$v^a = \begin{bmatrix} v \cdot x_a & v \cdot y_a & v \cdot z_a \end{bmatrix}^\top.$$

Let R_{ab} be the rotation matrix of frame Σ_b with respect to Σ_a. Then, the expressions v^a and v^b of a vector v in Σ_a and Σ_b are related by

$$v^a = R_{ab}v^b.$$

1. [25 points] This question is concerned with finding the degrees of freedom of linkages.

   (a) [10 points] Find the number of degrees of freedom for the spatial linkage below. This open- loop kinematic chain includes cylindrical, prismatic, and spherical joints.

   spatial mechanism

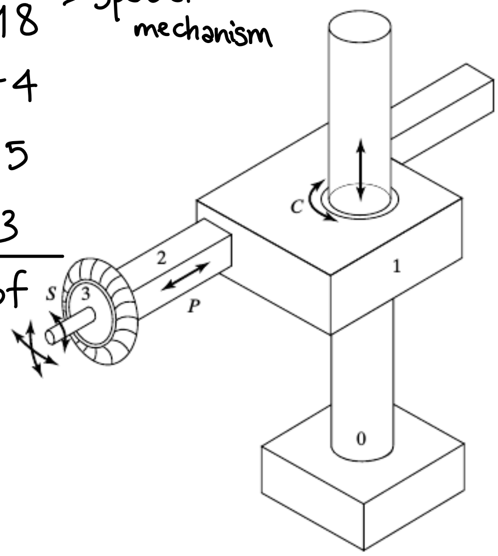
   3 rigid bodies: $3 \times 6 = 18$

   1 cylindrical joint: $-1 \times 4 = -4$

   1 prismatic joint: $-1 \times 5 = -5$

   1 spherical joint: $-1 \times 3 = -3$

   6 dof



   (b) [10 points] The linkage depicted below is a schematic representation of a piece of construction machinery. It has two hydraulic cylinders (links 1 and 2, and links 6 and 7), which may be considered cylindrical pairs. The other pairs are revolute joints. Find the number of degrees of freedom if this linkage is treated as a planar linkage.

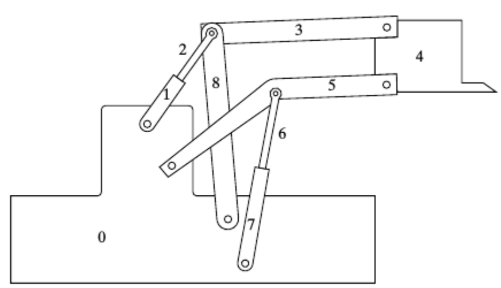
   planar mechanism

   8 rigid bodies: $8 \times 3 = 24$

   9 revolute joints: $-9 \times 2 = -18$

   2 prismatic joints: $-2 \times 2 = -4$

   2 dof

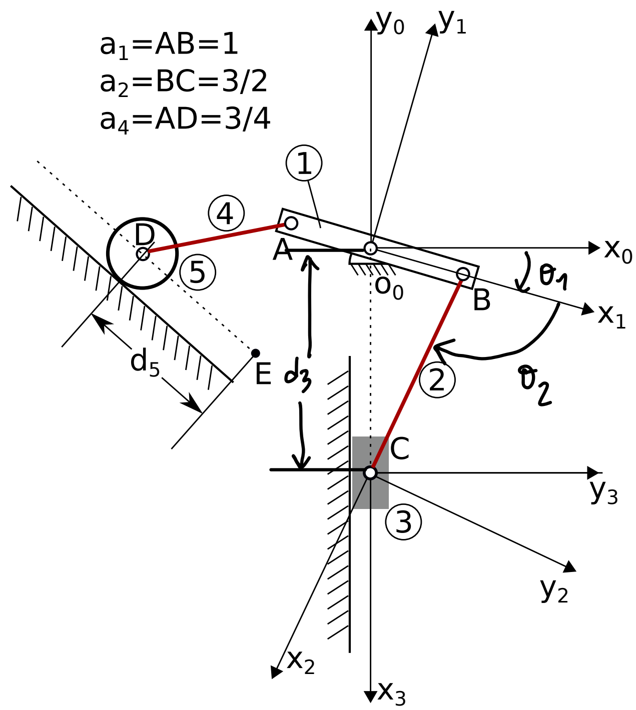


   (c) [5 points] Indicate the conditions that must be met for the planar linkage assumption to be valid in part (b).

   Planar motion applies whenever the motion occurs in a plane or in a set of parallel planes. In other words, the planes of motion of the links must all be parallel.

2. [35 points] Consider the linkage shown in the figure below. The origin of the world frame, o_0, is located in the middle of points A and B. We define the following quantities:

- θ_1: angle between the x_0– and x_1–axes.
- θ_2: angle between the x_1– and x_2–axes.,
- d_3: distance along the x_3–axis from the x_0–axis to the y_3–axis.
- θ_4: angle between the unit vector from D to E and the unit vector from D to A.



(a) [20 points] Find the rigid-body transformations between the world frame Σ_0 and the body-fixed coordinate frames Σ_1 through Σ_3 (established for you). Also find the rigid-body transformation between Σ_1 and Σ_3.

(b) [15 points] If $\theta_1 = -\frac{\pi}{9} = -20°$, compute the position vector from o_0 to C and express it in the world frame, Σ_0.

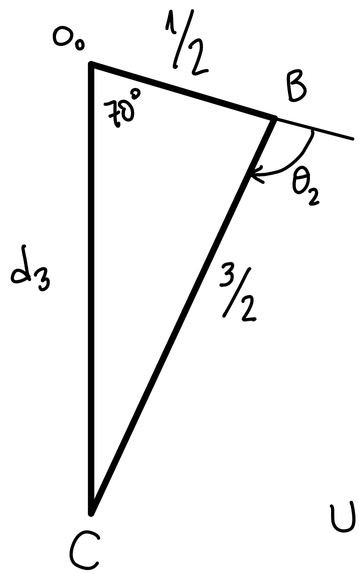
(a) $R_{01} = R_{z,\theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_{12} = R_{z,\theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$R_{02} = R_{01}R_{12} = R_{z,\theta_1+\theta_2} = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{03} = R_{z,-\pi/2} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_{13} = R_{01}^T R_{03} = R_{z,-(\theta_1+\pi/2)} = \begin{bmatrix} -s_1 & c_1 & 0 \\ -c_1 & -s_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_{01} = 0$, $P_{02} = P_{03} = \begin{bmatrix} 0 \\ -d_3 \\ 0 \end{bmatrix}^0$ $P_{13} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}^2 = R_{02}\begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2 c_{12} \\ a_2 s_{12} \\ 0 \end{bmatrix}^0$.

(b) Consider the following triangle and use law of cosines:



$$\left(\frac{3}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + d_3^2 - 2\frac{1}{2} d_3 \cos(70°)$$

or $\quad -2 + \cos(70°)\, d_3 + d_3^2 = 0,$

with the solution set given by

$$d_3 \in \{-1.254,\ 1.596\}.$$

Use law of cosines again:

$$d_3^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 2\,\frac{1}{2}\,\frac{3}{2}\cos(\theta_2)$$

$\underline{d_3 = -1.254}$

$\cos(\theta_2) = -0.619$

$\Rightarrow \theta_2 = 128.243°$

$$\boxed{\begin{array}{c} \underline{d_3 = 1.596} \\ \cos(\theta_2) = 0.030 \\ \theta_2 = -88.254° \end{array}}$$

3. [40 points] Consider the linkage shown in Figure 1. Let $d_1 = 59.5$ in. and $d_4 = 57$ in. be the vertical and horizontal distances from the origin of the world frame, Σ_0, to A and C, respectively. Furthermore, define the following angles:

   - θ_2: angle between x_0–axis and the unit vector from A to B.
   - θ_3: angle between x_0–axis and the unit vector from C to B.

   Other relevant quantities are presented within the figure.

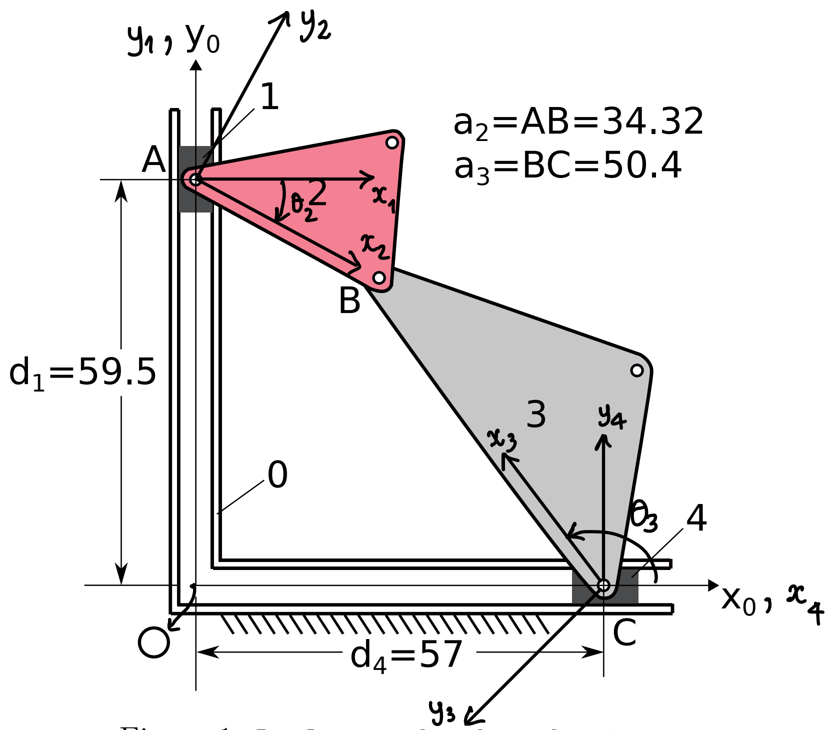


Figure 1: `Dual crosshead mechanism`

   (a) [5 points] Compute the mobility of this linkage.

   (b) [15 points] Establish coordinate frames fixed to each rigid body and compute the rigid-body transformation between the world frame (established on the figure) and each of these coordinate frames.

   (c) [20 points] Derive the loop equations in terms of d_1, θ_2, θ_3, and d_4 (the constants a_2 and a_3 are the only other quantities that may appear in these equations).

   (d) [8 points (bonus)] Solve the loop equations you to find the values of θ_2 and θ_3.

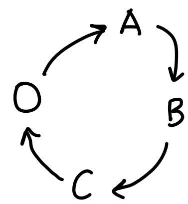
(a) 4 rigid bodies: $4 \times 3 = 12$
3 revolute joints: $-3 \times 2 = -6$
2 prismatic joints: $-2 \times 2 = -4$

2 dof.

(b) $R_{01} = R_{04} = I$; $R_{02} = R_{z,\theta_2}$, $R_{03} = R_{z,\theta_3}$.

$$p_{01} = p_{02} = \begin{bmatrix} 0 \\ d_1 \\ 0 \end{bmatrix}^0 ; \quad p_{03} = p_{04} = \begin{bmatrix} d_4 \\ 0 \\ 0 \end{bmatrix}^0$$

(c) Loop goes



: $p_{01} + p_{AB} + p_{BC} - p_{03} = 0$

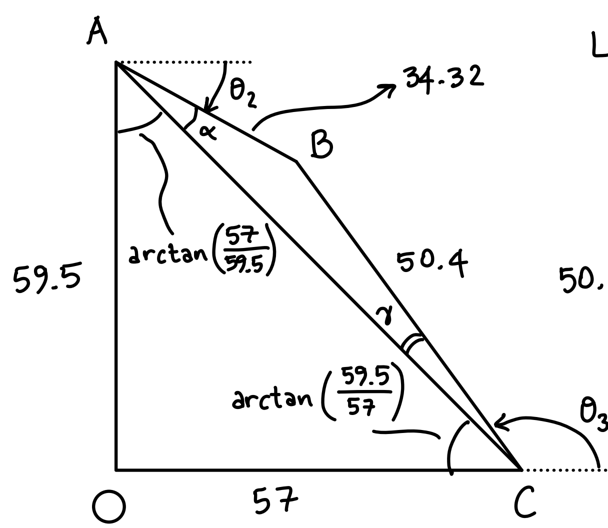
$$p_{AB} = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}^2 \Rightarrow p^0_{AB} = R_{02}\, p^2_{AB} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = a_2 \begin{bmatrix} c_2 \\ s_2 \\ 0 \end{bmatrix}^0$$

$$p_{BC} = \begin{bmatrix} -a_3 \\ 0 \\ 0 \end{bmatrix}^3 \Rightarrow p^0_{BC} = R_{03}\, p^3_{BC} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a_3 \\ 0 \\ 0 \end{bmatrix} = -a_3 \begin{bmatrix} c_3 \\ s_3 \\ 0 \end{bmatrix}^0$$

Equations are :

$$a_2 \cos(\theta_2) - a_3 \cos(\theta_3) - d_4 = 0$$
$$d_1 + a_2 \sin(\theta_2) - a_3 \sin(\theta_3) = 0 .$$

(d)



Let $\Delta = \sqrt{57^2 + 59.5^2} \approx 82.397$

Use law of cosines:

$$50.4^2 = 34.32^2 + 59.5^2 + 57^2 - 2(34.32)\Delta\cos(\alpha)$$

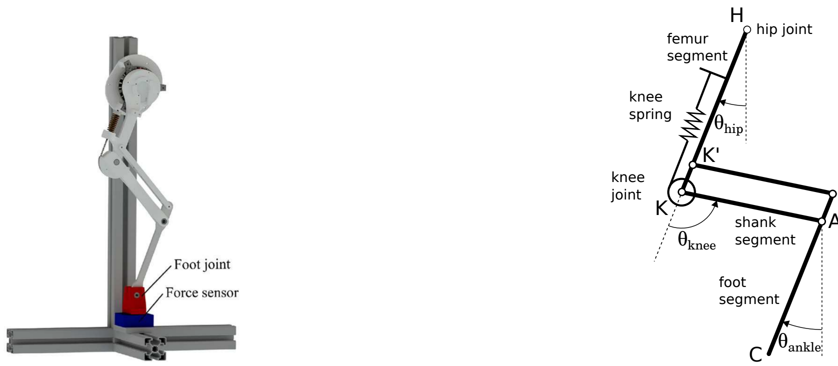
$$\Rightarrow \alpha = 16.352^\circ \Rightarrow |\theta_2| = 29.877^\circ$$

Since θ_2 is clockwise,

$$\boxed{\theta_2 = -29.877^\circ}$$

Similarly, from law of cosines: $34.32^2 = 50.4^2 + 59.5^2 + 57^2 - 2(50.4)\Delta\cos(\gamma)$

$$\Rightarrow \gamma = 11.053^\circ \Rightarrow \boxed{\theta_3 = 122.718^\circ}$$



(a) Rendered leg test stand with leg fixed into a rotary joint on hip and foot (red) on force sensor (blue).

(b) Schematic of cheetah-cub's spring-loaded leg mechanism.

Figure 2: `The leg mechanism of the cheetah-cub robot from EPFL.`

4. [20 points (bonus)] Figure 2 depicts the mechanism that acts as the legs of a quadruped robot, called Cheetah-cub, built in Switzerland at EPFL (see Figure 3). This mechanism mimics the leg of a real-world felidea (cat). It consists of three segments (links), the femur ($\overrightarrow{HK}$), the shank ($\overrightarrow{KA}$), and the foot ($\overrightarrow{AC}$).

   The fourbar linkage that the shank is comprised of ($K - K' - A' - A$) is a geometric parallelogram, keeping the femur and the foot segments parallel.

   (a) [5 points] Compute the mobility of the leg mechanism.

   (b) [15 points] Place coordinate system, Σ_{hip}, at the hip (H) and compute the translation of the tip of the foot (C) with respect to this coordinate system when $\theta_{\text{hip}} = -15°$ and $\theta_{\text{knee}} = 105°$, assuming that the lengths of all segments are equal to $\ell = 150$mm.

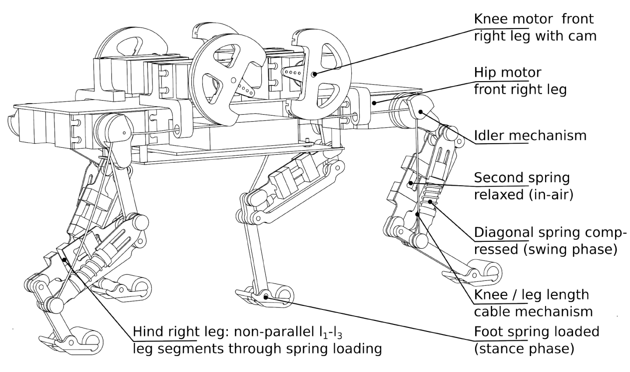


Figure 3: `The Cheetah-cub robot from EPFL.` The robot's view from hind right; its front is to the right. Front right and hind left leg are in swing phase, front left and hind right are in stance phase.

(a) There are 4 moving links: 1 femur, 2 shank, 1 foot segments.

There are 5 revolute joints.

$$\begin{aligned} 4 \times 3 &= 12 \\ -5 \times 2 &= -10 \\ \hline &\ 2 \text{ dof.} \end{aligned}$$

(b) $\overrightarrow{HK} = -\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix}^0$, $\overrightarrow{KA} = -\ell \begin{bmatrix} -\sin(\theta_{hip} + \theta_{knee}) \\ \cos(\theta_{hip} + \theta_{knee}) \end{bmatrix}^0$

$$\overrightarrow{AC} = -\ell \begin{bmatrix} -\sin(\theta_{ankle}) \\ \cos(\theta_{ankle}) \end{bmatrix}^0 = -\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix}^0$$

femur and foot segments are parallel.

$$\begin{aligned} \overrightarrow{HC} &= \overrightarrow{HK} + \overrightarrow{KA} + \overrightarrow{AC} \\ &= -2\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix} - \ell \begin{bmatrix} -\sin(\theta_{hip} + \theta_{knee}) \\ \cos(\theta_{hip} + \theta_{knee}) \end{bmatrix} = \begin{bmatrix} 28.42 \\ -183.71 \end{bmatrix} \text{ mm.} \end{aligned}$$