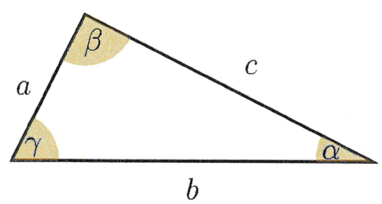
ME 380: Kinematics and Machine Dynamics
Fall 2020 / Midterm Examination #1

Name: Solution Manual

Angle sum and difference identities

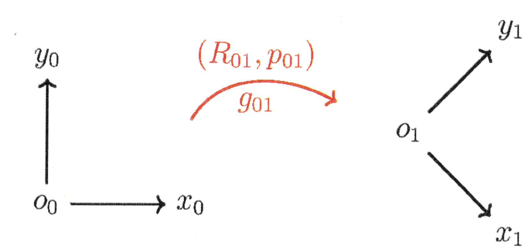
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Rigid Body Transformations



$\mathbb{R}^3 \ni p_{01}$: position vector from o_0 to o_1.

$$\mathbb{R}^{3\times 3} \ni R_{01} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$g_{01} = \left[\begin{array}{c|c} R_{01} & p_{01} \\ \hline 0 & 1 \end{array}\right]$$

Composition rules: $R_{ac} = R_{ab}R_{bc}$ and $g_{ac} = g_{ab}g_{bc}$.

Vector Operations

Let v and w be vectors in $\mathbb{R}^3$, then the inner (dot) product of v and w is defined by

$$v \cdot w = \|v\|\,\|w\|\cos\theta,$$

where θ is the angle between v and w.

Let Σ_a be a coordinate frame, and v be a vector in $\mathbb{R}^3$. Then, the expression of v in the coordinate frame Σ_a is given by

$$v^a = \begin{bmatrix} v \cdot x_a & v \cdot y_a & v \cdot z_a \end{bmatrix}^\top.$$

Let R_{ab} be the rotation matrix of frame Σ_b with respect to Σ_a. Then, the expressions v^a and v^b of a vector v in Σ_a and Σ_b are related by

$$v^a = R_{ab}v^b.$$

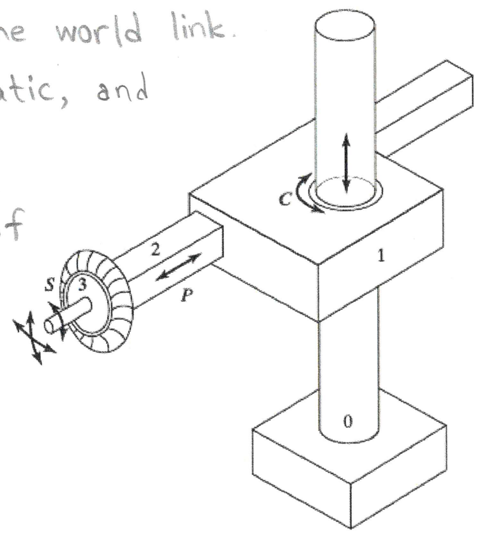
1. [25 points] This question is concerned with finding the degrees of freedom of linkages.

(a) [10 points] Find the number of degrees of freedom for the spatial linkage below. This open- loop kinematic chain includes cylindrical, prismatic, and spherical joints.

There are 4 links, including the world link.
There are 1 cylindrical, 1 prismatic, and 1 spherical joint.

$6(4-1)$ → unconstrained dof
$-4(1)$ → cylindrical
$-5(1)$ → prismatic
$-3(1)$ → spherical

6 DoF

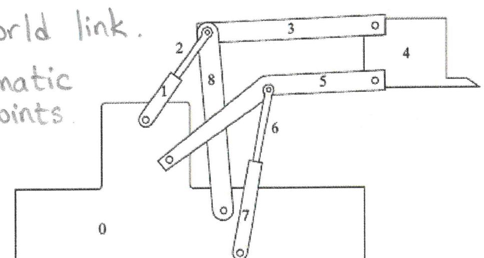


(b) [10 points] The linkage depicted below is a schematic representation of a piece of construction machinery. It has two hydraulic cylinders (links 1 and 2, and links 6 and 7), which may be considered cylindrical pairs. The other pairs are revolute joints. Find the number of degrees of freedom if this linkage is treated as a planar linkage.

There are 9 links, including the world link.
There are 9 revolute and 2 prismatic joints.

$3(9-1)$ → unconstrained dof
$-2(9)$ → revolute
$-2(2)$ → prismatic

2 DoF



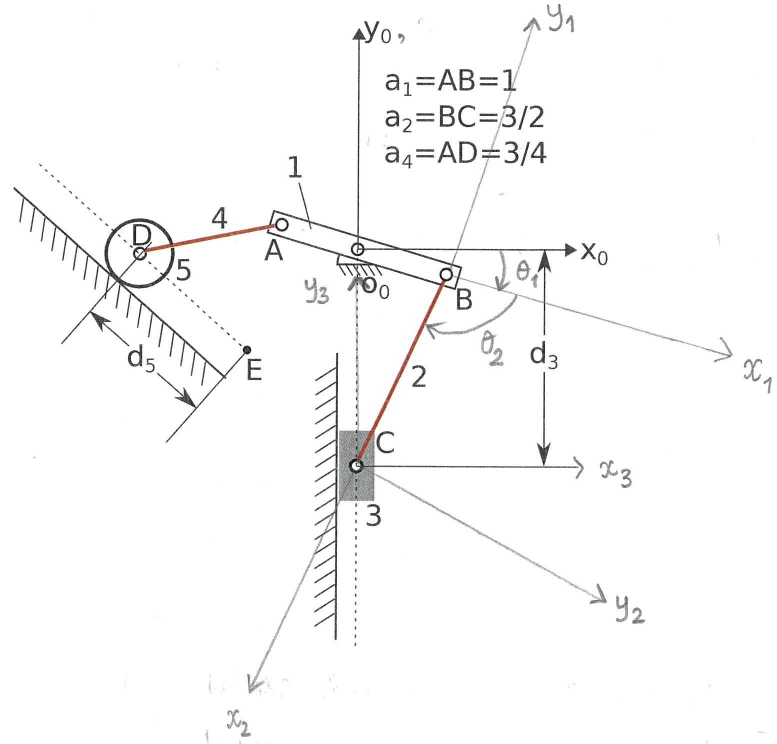
(c) [5 points] Indicate the conditions that must be met for the planar linkage assumption to be valid in part (b).

Planar motion applies if the motion occurs in a plane or in a set of parallel planes. In other words, the planes of motion of the links must all be parallel.

2. [35 points] Consider the linkage shown in the figure below. Note that the origin of the world frame, o_0, is located exactly in the middle of points A and B. We define the following angles:

- θ_1: angle between the x_0-axis and the unit vector from o_0 to B.
- θ_2: angle between the unit vector from o_0 to B and the unit vector from B to C.
- θ_4: angle between the unit vector from D to E and the unit vector from D to A.

Please notice some other quantities that are presented within the figure.



(a) [20 points] Establish coordinate frames on each of the links 1 through 3. Find the rigid-body transformations (translation and rotation) between the world frame (established for you) and the body-fixed coordinate frames Σ_1 through Σ_3.

(b) [15 points] If $\theta_1 = -\frac{\pi}{9} = -20°$, compute the position vector from o_0 to C and express it in the world frame, Σ_0.

(a) Coordinate frames are established on the figure.

$R_{01} = R_{z,\theta_1}$, $R_{02} = R_{z,\theta_1+\theta_2}$, $R_{03} = I$, where $R_{z,\varphi} = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P^0_{01} = \frac{a_1}{2}\begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}$, $P^0_{12} = a_2 \begin{bmatrix} c_{12} \\ s_{12} \\ 0 \end{bmatrix}$, $P^0_{03} = -d_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) The loop equations, written in Σ_0, are:

$$\frac{1}{2}\cos(\theta_1) + \frac{3}{2}\cos(\theta_1+\theta_2) = 0,$$
$$\frac{1}{2}\sin(\theta_1) + \frac{3}{2}\sin(\theta_1+\theta_2) + d_3 = 0.$$

Express the loop equations as
$$\frac{3}{2}\cos(\theta_1+\theta_2) = -\frac{\cos(\theta_1)}{2}$$
$$\frac{3}{2}\sin(\theta_1+\theta_2) = -d_3 - \frac{\sin(\theta_1)}{2} \qquad (*)$$

Square both equations in $(*)$ and sum them to get

$$-2 + \sin(\theta_1)d_3 + d_3^2 = 0 \quad \xrightarrow[\text{for } d_3]{\text{solve}} \quad d_3 = \frac{1}{2}\left(-\sin(\theta_1) \pm \sqrt{8+\sin(\theta_1)}\right) = \{-1.254,\ 1.596\}.$$

Now that we know d_3, we insert it in $(*)$ after we divide the lower equation by the upper equation:

$$\tan(\theta_1+\theta_2) = \frac{-d_3 - \frac{\sin(\theta_1)}{2}}{-\frac{\cos(\theta_1)}{2}} \Rightarrow \theta_2 = \operatorname{atan}_2\left(-d_3 - \frac{\sin(\theta_1)}{2},\ \frac{-\cos(\theta_1)}{2}\right) - \theta_1$$
$$= \{2.238,\ -1.540\} \text{ rad}.$$
$$\approx \{128.25°,\ -88.25°\}.$$

The vector from o_0 to C is equal to p_{02} or p_{03}; from either we get

$$p_{02} = \begin{bmatrix} \frac{1}{2}\cos(-\pi/9) + \frac{3}{2}\cos(-\frac{\pi}{9} - 1.540) \\ \frac{1}{2}\sin(-\pi/9) + \frac{3}{2}\sin(-\pi/9 - 1.540) \end{bmatrix} = \begin{bmatrix} 0 \\ -1.596 \end{bmatrix}.$$

$$p_{03} = \begin{bmatrix} 0 \\ -d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.596 \end{bmatrix}$$

Similarly, for the other d_3 and θ_2 pair.

3. [40 points] Consider the linkage shown in Figure 1. Let $d_1 = 59.5$ in. and $d_4 = 57$ in. be the vertical and horizontal distances from the origin of the world frame, Σ_0, to A and B, respectively. Furthermore, define the following angles:

- θ_2: angle between x_0 axis and the unit vector from A to B.
- θ_3: angle between x_0 axis and the unit vector from C to B.

Other relevant quantities are presented within the figure.

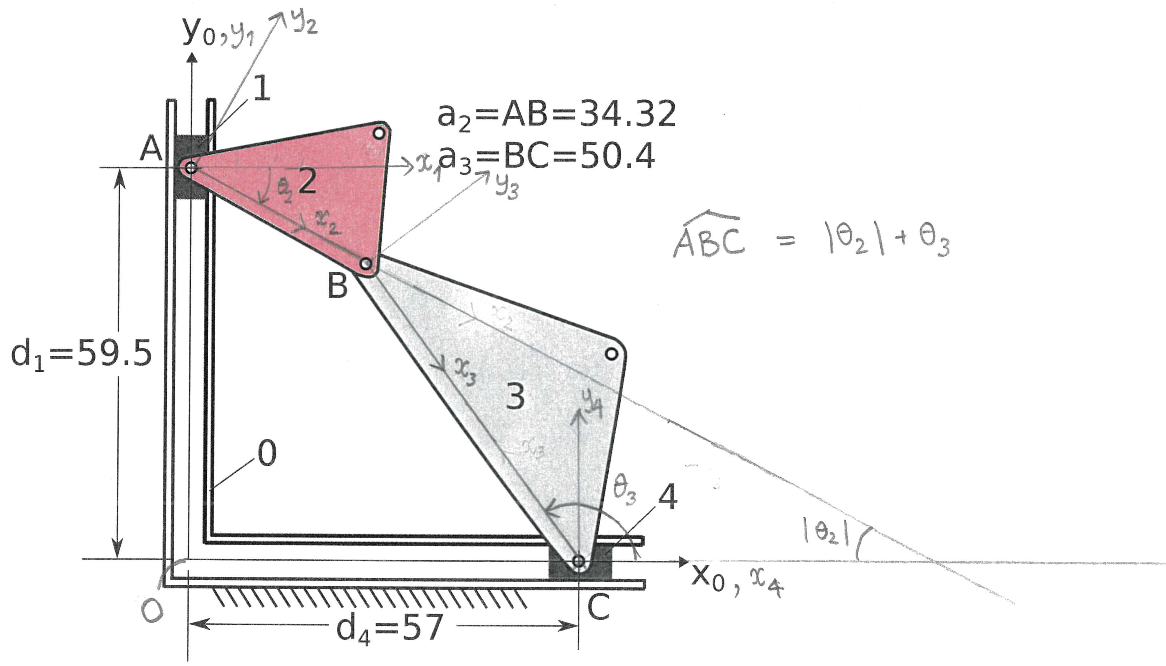


Figure 1: `Dual crosshead mechanism`

Answer the following questions about this mechanism.

(a) [5 points] Compute the mobility of this linkage.

(b) [15 points] Establish coordinate frames fixed to each rigid body and compute the transformation (translation and rotation) between the world frame (established on the figure) and each of these coordinate frames.

(c) [20 points] Derive the loop equations in terms of d_1, θ_2, θ_3, and d_4.

(d) [8 points (bonus)] Solve the loop equations you derived in part (c) to find the values of θ_2 and θ_3.

(a) There are 5 links, including the world link.
There are 3 revolute joints and 2 prismatic joints:

$$3\cdot(5-1) - 2(3) - 2(2) = 2 \text{ DoF}.$$

(b) Define Σ_1 and Σ_4 to have the same orientation as Σ_0, with origins o_1 and o_4 at A and C, respectively.

Place the origins o_2 of Σ_2 at A and o_3 of Σ_3 at B. Let x_2 point along $\overrightarrow{AB}$ and x_3 along $\overrightarrow{BC}$. Let $z_0 = z_2 = z_3$.

We have $R_{01} = R_{03} = I$, $p^0_{01} = d_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $p^0_{04} = d_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

while $R_{02} = R_{z,\theta_2}$ and $R_{03} = R_{z,\theta_3 - \pi}$. Furthermore, $p_{02} = p_{01}$,

$p_{03} = p_{02} + p_{23}$, where $p^0_{23} = a_2 \begin{bmatrix} c_2 \\ s_2 \\ 0 \end{bmatrix}$. Hence $p^0_{03} = \begin{bmatrix} a_2 c_2 \\ d_1 + a_2 s_2 \\ 0 \end{bmatrix}$.

(c) Loop equation: $p_{02} + p_{34} - p_{04} = 0$.

We have yet to find p_{34} in Σ_0: $p_{34} = a_3 R_{03} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -a_3 \begin{bmatrix} c_3 \\ s_3 \\ 0 \end{bmatrix}$.

Hence, in Σ_0, the loop equations are given by

$$a_2 \cos(\theta_2) - a_3 \cos(\theta_3) - d_4 = 0 \quad (1a)$$

$$d_1 + a_2 \sin(\theta_2) - a_3 \sin(\theta_3) = 0 \quad (1b)$$

(d) By some elementary Euclidean geometry, we see that the angle $\widehat{ABC}$ is equal to $|\theta_2| + \theta_3$ (see figure). Apply law of cosines to $\triangle ABC$:

$$d_1^2 + d_4^2 = a_2^2 + a_3^2 - 2 a_2 a_3 \cos(|\theta_2| + \theta_3) \Rightarrow |\theta_2| + \theta_3 = 2.663 \approx 152.3^\circ.$$

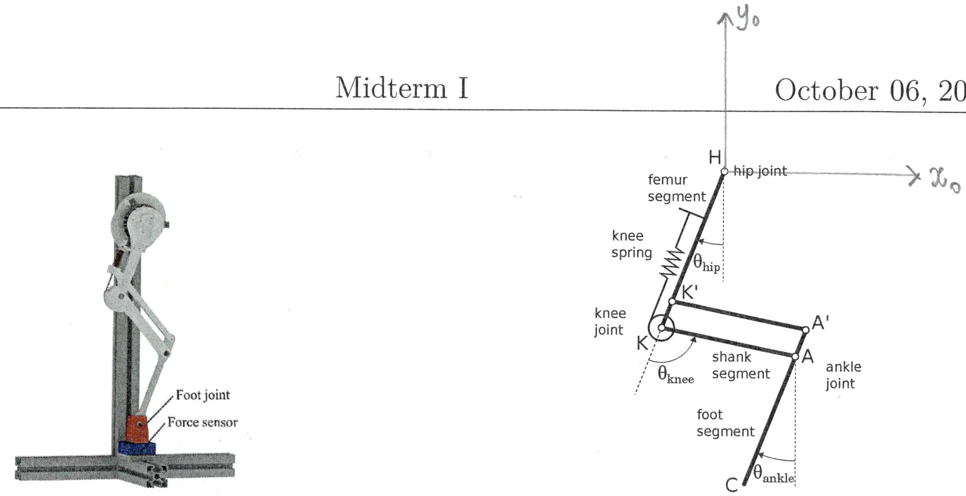
We can compute the angle $\widehat{CAB} = \frac{\pi}{2} - \widehat{CAO} - |\theta_2|$; but $\widehat{CAO} = \arctan\left(\frac{d_4}{d_1}\right) = 0.764$, so that $\widehat{CAB} = 0.807 - |\theta_2|$.

Now, use law of cosines again on $\triangle ABC$:

$$a_3^2 = a_2^2 + d_1^2 + d_4^2 - 2 a_2 \sqrt{d_1^2 + d_4^2} \cos(0.807 - |\theta_2|) \Rightarrow |\theta_2| = 0.521 \approx 29.87^\circ.$$

But, we know (see figure) $\theta_2 < 0$, so $\theta_2 = -0.521 \approx -29.87^\circ$.

Moreover, $\theta_3 = 2.663 - 0.521 = 2.142$ rad $\approx 122.7^\circ$.



(a) Rendered leg test stand with leg fixed into a rotary joint on hip and foot (red) on force sensor (blue).

(b) Schematic of cheetah-cub's spring-loaded leg mechanism.

Figure 2: `The leg mechanism of the cheetah-cub robot from EPFL.`

4. [20 points (bonus)] Figure 2 depicts the mechanism that acts as the legs of a quadruped robot, called Cheetah-cub, built in Switzerland at EPFL (see Figure 3). This mechanism mimics the leg of a real-world felidea (cat). It consists of three segments (links), the femur ($\overrightarrow{HK}$), the shank ($\overrightarrow{KA}$), and the foot ($\overrightarrow{AC}$).

   The fourbar linkage that the shank is comprised of ($K - K' - A' - A$) is a geometric parallelogram, keeping the femur and the foot segments parallel.

   (a) [5 points] Compute the mobility of the leg mechanism.

   (b) [15 points] Place coordinate system, Σ_{hip}, at the hip (H) and compute the translation of the tip of the foot (C) with respect to this coordinate system when $\theta_{\text{hip}} = -15°$ and $\theta_{\text{knee}} = 105°$, assuming that the lengths of all segments are equal to $\ell = 150$mm.

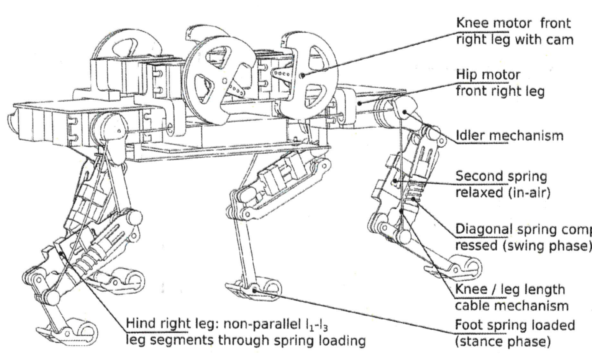


Figure 3: `The Cheetah-cub robot from EPFL.` The robot's view from hind right; its front is to the right. Front right and hind left leg are in swing phase, front left and hind right are in stance phase.

(a) There are 4 moving links: 1 femur segment,
2 shank segments,
1 foot segment.

There are 5 revolute joints:

$$\begin{array}{r} 3 \cdot 4 \\ - 2 \cdot 5 \\ \hline 2 \end{array}$$ degrees of freedom.

(b) $$\overrightarrow{HK}^0 = -\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix}^0$$

$$\overrightarrow{KA}^0 = -\ell \begin{bmatrix} -\sin(\theta_{hip} + \theta_{knee}) \\ \cos(\theta_{hip} + \theta_{knee}) \end{bmatrix}^0$$

$$\overrightarrow{AC}^0 = -\ell \begin{bmatrix} -\sin(\theta_{ankle}) \\ \cos(\theta_{ankle}) \end{bmatrix}^0 = -\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix}^0$$

femur and foot segments are parallel.

$$\overrightarrow{HC} = \overrightarrow{HK} + \overrightarrow{KA} + \overrightarrow{AC} = -2\ell \begin{bmatrix} -\sin(\theta_{hip}) \\ \cos(\theta_{hip}) \end{bmatrix} - \ell \begin{bmatrix} -\sin(\theta_{hip} + \theta_{knee}) \\ \cos(\theta_{hip} + \theta_{knee}) \end{bmatrix}$$

$$= \begin{bmatrix} 28.42 \\ -183.71 \end{bmatrix} \text{mm.}$$