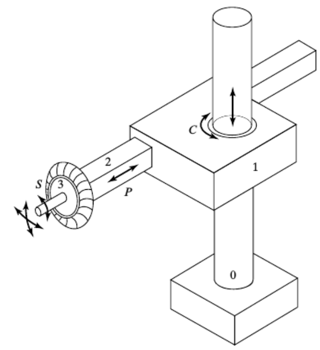
ME 380
Kinematics and Machine Dynamics
Midterm Solutions

1. [25 points] This question is concerned with finding the degrees of freedom of linkages (appears in p. 98 of Wilson and Sadler).

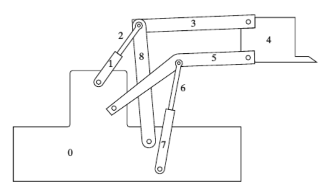
   (a) [10 points] Find the number of degrees of freedom for the spatial linkage below. This open- loop kinematic chain includes cylindrical, prismatic, and spherical joints.

   > **Solution:** There are 4 links, including the fixed link.
   > There are 1 cylindrical, 1 prismatic, and 1 spherical joints.
   >
   > $$6(4-1)-4(1)-5(1)-3(1)=6 \text{ DoF}$$



   (b) [10 points] The linkage depicted below is a schematic representation of a piece of construction machinery. It has two hydraulic cylinders (links 1 and 2, and links 6 and 7), which may be considered cylindrical pairs. The other pairs are revolute joints. Find the number of degrees of freedom if this linkage is treated as a planar linkage.

   > **Solution:** There are 9 links, including the fixed link.
   > There are 9 revolute joints, and 2 prismatic joints.
   >
   > $$3(9-1)-2(9)-2(2)=2 \text{ DoF}$$



   (c) [5 points] Indicate the conditions that must be met for the planar linkage assumption to be valid in part (b).

   > **Solution:** The planar motion applies if the motion occurs in a plane or in a set of parallel planes. The planes of motion of the links must all be parallel.

2. [35 points] Consider the quick-return shaper mechanism shown in Figure 1. Excluding the fixed link, it consists of 5 links, one of which is the slider, as labelled in the figure. Let l_k denote the length of link $k \in \{1,3,4\}$.

   (a) [7 points] If link 1 is the input and link 5 is the output, state which link is the coupler. Also, clearly mark the transmission angle by the quantity φ.

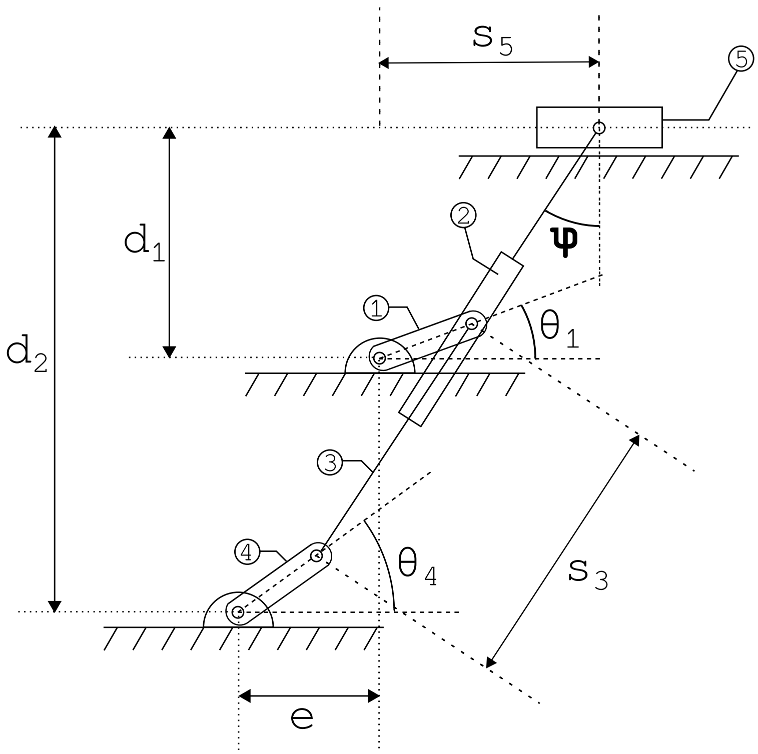


Figure 1: `Quick-return shaper mechanism`

> **Solution:** The transmission angle is marked on the Figure 1 with a bold font. The coupler is the link 3.

(b) [18 points] Analytically derive the loop equations in terms of the variables θ_1, θ_4, s_3, s_5, the transmission angle φ, and the various constants d_1, d_2, e, l_1, l_3, and l_4.

> **Solution:**
>
> $$l_1 \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} - s_3 \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix} - l_4 \begin{bmatrix} \cos\theta_4 \\ \sin\theta_4 \end{bmatrix} + \begin{bmatrix} e \\ d_2 - d_1 \end{bmatrix} = 0, \tag{1a}$$
>
> $$l_1 \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + (l_3 - s_3) \begin{bmatrix} \sin\varphi \\ \cos\varphi \end{bmatrix} - \begin{bmatrix} s_5 \\ d_1 \end{bmatrix} = 0. \tag{1b}$$

(c) [10 points] Set up an optimization problem, whose solution yields the minimum length $l_{3_{min}}$ of link 3, such that the transmission angle satisfies $\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$.

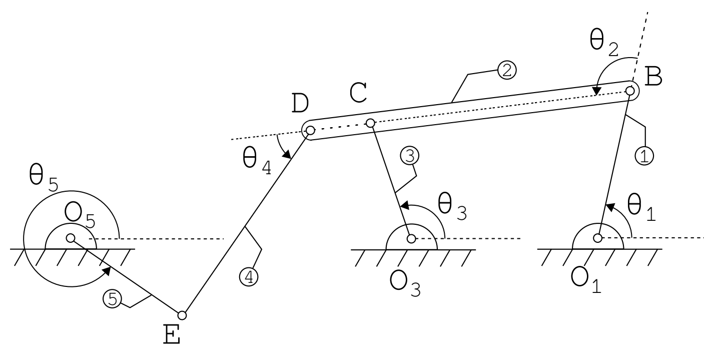
> **Solution:**
>
> $$\text{minimize} \qquad l_3$$

$$\begin{aligned} \text{subject to} \quad & (1) \\ \frac{\pi}{4} &\leq \varphi \leq \frac{3\pi}{4} \\ s_3 &\leq l_3 \end{aligned}$$

Turns out the answer is $l_{3_{min}} = 4.24264$, with the remaining variables given by $\left[\theta_1^*, \theta_4^*, s_3^*, s_5^*, \varphi^*\right] = \left[1.1476, \frac{\pi}{2}, 2.7032, 1.5000, \frac{\pi}{4}\right]$.

3. [40 points] Consider the double four-bar mechanism shown in the figure below. Excluding the fixed link, it consists of 5 links, all connected together with revolute joints(appears in p. 263 of Wilson and Sadler).

Quantity	Endpoints	Length [in]
l_0	$\lvert O_3O_1\rvert$	1
l_0'	$\lvert O_5O_3\rvert$	4
l_1	$\lvert O_1B\rvert$	3
l_2	$\lvert BC\rvert$	2.25
l_2'	$\lvert BD\rvert - l_2$	1
l_3	$\lvert O_3C\rvert$	2
l_4	$\lvert DE\rvert$	2.75
l_5	$\lvert O_5E\rvert$	1



(a) [8 points] Derive the position-level loop equation for this mechanism.

Solution: Let $p_0 = \overrightarrow{O_3O_1}$, $p_0' = \overrightarrow{O_5O_3}$, $p_1 = \overrightarrow{O_1B}$, $p_2 = \overrightarrow{BC}$, $p_2' = \overrightarrow{CD}$, $p_3 = \overrightarrow{O_3C}$, $p_4 = \overrightarrow{DE}$, and $p_5 = \overrightarrow{O_5E}$.

$$p_0 + p_1 + p_2 - p_3 = l_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + l_1 \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + l_2 \begin{bmatrix} \cos\theta_{12} \\ \sin\theta_{12} \end{bmatrix} - l_3 \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \end{bmatrix} = 0, \quad (2a)$$

$$\begin{aligned} & p_3 + p_2' + p_4 - p_5 + p_0' = \\ & = l_0' \begin{bmatrix} 1 \\ 0 \end{bmatrix} + l_3 \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \end{bmatrix} + l_2' \begin{bmatrix} \cos\theta_{12} \\ \sin\theta_{12} \end{bmatrix} + l_4 \begin{bmatrix} \cos\theta_{124} \\ \sin\theta_{124} \end{bmatrix} - l_5 \begin{bmatrix} \cos\theta_5 \\ \sin\theta_5 \end{bmatrix} = 0, \end{aligned} \quad (2b)$$

where $\theta_{ij} := \theta_i + \theta_j$ and $\theta_{ijk} := \theta_i + \theta_j + \theta_k$.

(b) [16 points] Use vector cross product method to solve for the position-level kinematics at an instant when $\theta_1 = \frac{5\pi}{12}$ rad (solve for θ_2, θ_3, θ_4, and θ_5).

Solution: Using case 3 with the loop BCO_3 oriented counter clockwise and the loop DEO_5 oriented clockwise yields the following

$p_{d1} = p_0 + p_1$	$\overrightarrow{O_3B}$	$\begin{bmatrix}1.7764 & 2.8978\end{bmatrix}$	–
p_2	$\overrightarrow{BC}$	$\begin{bmatrix}-2.0546 & -0.9172\end{bmatrix}$	$\theta_2 = 2.2525$
p_3	$\overrightarrow{O_3C}$	$\begin{bmatrix}-0.2781 & 1.9806\end{bmatrix}$	$\theta_3 = 1.7103$
$p_2' = l_2' \frac{p_2}{\|p_2\|}$	$\overrightarrow{CD}$	$\begin{bmatrix}-0.9131 & -0.4076\end{bmatrix}$	–
$p_{d2} = p_0' + p_3 + p_2'$	$\overrightarrow{O_5D}$	$\begin{bmatrix}2.8088 & 1.5729\end{bmatrix}$	–
p_4	$\overrightarrow{DE}$	$\begin{bmatrix}-1.8993 & -1.9887\end{bmatrix}$	$\theta_4 = 0.3885$
p_5	$\overrightarrow{O_5E}$	$\begin{bmatrix}0.9094 & -0.4158\end{bmatrix}$	$\theta_5 = -0.4289$

(c) [16 points] At this instant, if ${}^0\omega^1 = 10\hat{e}_3 \frac{rad}{s}$, what is ${}^0\omega^5$?

Solution: To solve this question, we need to write down the velocity-level kinematic equations. To do that, we need the position level kinematic equations that we derived in part (a). Differentiating (2) yields the velocity-level kinematic equations

$$\begin{aligned}
&{}^0\omega^1 \times p_1 + {}^0\omega^2 \times p_2 - {}^0\omega^3 \times p_3 \ = \\
&= l_1\dot{\theta}_1 \begin{bmatrix}-\sin\theta_1 \\ \cos\theta_1\end{bmatrix} + l_2\dot{\theta}_{12} \begin{bmatrix}-\sin\theta_{12} \\ \cos\theta_{12}\end{bmatrix} - l_3\dot{\theta}_3 \begin{bmatrix}-\sin\theta_3 \\ \cos\theta_3\end{bmatrix} = 0,
\end{aligned} \tag{3a}$$

$$\begin{aligned}
&{}^0\omega^3 \times p_3 + {}^0\omega^2 \times p_2' + {}^0\omega^4 \times p_4 - {}^0\omega^5 \times p_5 \ = \\
&= l_3\dot{\theta}_3 \begin{bmatrix}-\sin\theta_3 \\ \cos\theta_3\end{bmatrix} + l_2'\dot{\theta}_{12} \begin{bmatrix}-\sin\theta_{12} \\ \cos\theta_{12}\end{bmatrix} + l_4\dot{\theta}_{124} \begin{bmatrix}-\sin\theta_{124} \\ \cos\theta_{124}\end{bmatrix} - l_5\dot{\theta}_5 \begin{bmatrix}-\sin\theta_5 \\ \cos\theta_5\end{bmatrix} = 0.
\end{aligned} \tag{3b}$$

where $\dot{\theta}_1 = {}^0\omega^1$, $\dot{\theta}_{12} := \dot{\theta}_1 + \dot{\theta}_2 = {}^0\omega^2$, $\dot{\theta}_3 = {}^0\omega^3$, $\dot{\theta}_{124} := \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_4 = {}^0\omega^4$, and $\dot{\theta}_5 = {}^0\omega^5$. Define $\omega := \begin{bmatrix}\dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 & \dot{\theta}_5\end{bmatrix}^\top$. Writing out equations (3) in matrix form gives an equation of the form $J\omega = f$. We have

$$J \ = \begin{bmatrix}J_1 & 0 \\ J_2 & J_3\end{bmatrix} := \begin{bmatrix}-l_2\sin\theta_{12} & l_3\sin\theta_3 & 0 & 0 \\ l_2\cos\theta_{12} & -l_3\cos\theta_3 & 0 & 0 \\ -l_2'\sin\theta_{12} & -l_3\sin\theta_3 & -l_4\sin\theta_{124} & l_5\sin\theta_5 \\ l_2'\cos\theta_{12} & l_3\cos\theta_3 & l_4\cos\theta_{124} & -l_5\cos\theta_5\end{bmatrix}$$

$$f \ = \begin{bmatrix}f_1 \\ f_2\end{bmatrix} = \begin{bmatrix}l_1\sin\theta_1 + l_2\sin\theta_{12} \\ -(l_1\cos\theta_1 + l_2\cos\theta_{12}) \\ 0 \\ 0\end{bmatrix}\dot{\theta}_1$$

where J_1 is the top-left, J_2 is the bottom-left and J_3 is the bottom-right 2-by-2 submatrices of J. Similarly, f_1 denotes the top 2 components and f_2

denotes the bottom two components of the vector f. Also, let $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$, where $\omega_1 = \begin{bmatrix} \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^\top$ and $\omega_2 = \begin{bmatrix} \dot{\theta}_4 & \dot{\theta}_5 \end{bmatrix}^\top$.
Since there J has a block-diagonal structure, we can solve the linear equation $J\omega = f$ in two-steps where each step involves solving for 2 variables. That is, we first solve the equation $J_1\omega_1 = f_1$ and then use this solution to solve $J_2\omega_1 + J_3\omega_2 = 0$, yielding

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} J_1^{-1} \\ -J_3^{-1}J_2J_1^{-1} \end{bmatrix} f_1 = \begin{bmatrix} J_1^{-1} & 0 \\ -J_3^{-1}J_2J_1^{-1} & J_3^{-1} \end{bmatrix} f.$$

Performing this operation for ${}^0\omega^1 = \dot{\theta}_1 = 10 \, \frac{rad}{s}$ yields ${}^0\omega^5 = \dot{\theta}_5 = 2.6625 \, \frac{rad}{s}$.

4. [15 points (bonus)] Consider the planar two-link manipulator depicted in Figure 2. Suppose that the length l_1 of link 1 is unity.

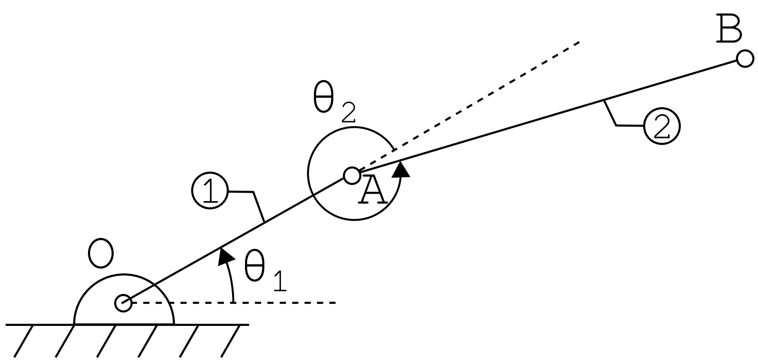


Figure 2: `Planar two-link manipulator`

A point Q moves from A towards B along link 2 with a constant rate 1 $\frac{m}{s}$ while link 1 rotates from $\theta_1(0) = \frac{\pi}{6}$ to $\theta_1(3) = \frac{2\pi}{3}$ at a constant rate in 3 seconds. During this time, link 2 rotates from $\theta_2(0) = -\frac{\pi}{12}$ to $\theta_2(3)$ to $\theta_2(3) = -\frac{\pi}{3}$ at a constant rate. Find the position of point Q as a function of time, expressed in the world frame. What is its position at $t = 2.5$ seconds?

Solution: Suppose the point Q is a distance l_Q away from the point A at some point in time. Then, the position vector p_Q from the point O to the point Q is given by

$$p_Q = l_1 R_{\theta_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + l_Q R_{\theta_1+\theta_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{4}$$

where R_ϕ is the rotation matrix along the axis perpendicular to the plane by an amount ϕ. We are given that

$$\begin{aligned}
\theta_1(t) &= \frac{\pi}{6}(1+t) \quad \Longrightarrow \quad \theta_1(0) = \frac{\pi}{6}, \quad \theta_1(3) = \frac{2\pi}{3}, \\
\theta_2(t) &= -\frac{\pi}{12}(1+t) \quad \Longrightarrow \quad \theta_2(0) = -\frac{\pi}{12}, \quad \theta_2(3) = -\frac{\pi}{3}, \\
l_Q(t) &= t.
\end{aligned}$$

Plugging these into equation (4) yields

$$p_Q(t) = l_1 \begin{bmatrix} \cos\left(\frac{\pi}{6}(1+t)\right) \\ \sin\left(\frac{\pi}{6}(1+t)\right) \end{bmatrix} + t \begin{bmatrix} \cos\left(\frac{\pi}{12}(1+t)\right) \\ \sin\left(\frac{\pi}{12}(1+t)\right) \end{bmatrix}.$$

Evaluating this at $t = 2.5$ gives

$$p_Q(2.5) = \begin{bmatrix} 1.26308 \\ 2.94931 \end{bmatrix}.$$