ME 380
Kinematics and Machine Dynamics
Linear Algebra Worksheet

1. Carry out the following matrix multiplications using the various different techniques we went over in class.

   (a) Calculate Ax, where

$$A = \begin{bmatrix} 12 & 17 & -8 & -5 & -6 \\ 3 & 12 & -19 & -3 & -14 \\ -8 & 16 & 14 & -13 & 16 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

Solution: $Ax = \begin{bmatrix} 82 & 24 & 49 \end{bmatrix}^\top$.

   (b) Calculate AB, where

$$A = \begin{bmatrix} -17 & -7 \\ 4 & -13 \\ 12 & -14 \\ -17 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 13 & -12 & 13 & 13 & -1 & 8 \\ -16 & 16 & 17 & -11 & 16 & -13 & -6 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 146 & -333 & 85 & -144 & -333 & 108 & -94 \\ 200 & -156 & -269 & 195 & -156 & 165 & 110 \\ 200 & -68 & -382 & 310 & -68 & 170 & 180 \\ -78 & -109 & 323 & -298 & -109 & -74 & -178 \end{bmatrix}$$

2. Solve the following systems of equations using matrix algebra. If more than one solution exist, find all the solutions. If no solution exists, explain why, and find an x such that $\|Ax - b\|$ is minimized.

   (a)

$$\begin{aligned} -2x_1 + 18x_2 &= -0.48 \\ 7x_1 - 8x_2 &= 0.04 \end{aligned}$$

Solution: The matrix A is invertible, so the solution can be found by $x = A^{-1}b$: $x_1 = -0.028363636$, $x_2 = -0.029818181818$

(b)

$$\begin{aligned} -13x_1 - 18x_2 - 3x_3 - 17x_4 &= 35 \\ 20x_1 - 6x_2 - 15x_3 - 18x_4 &= -57 \end{aligned}$$

Solution: Any vector of the form $x = x_p + \alpha_1 x_{h_1} + \alpha_2 x_{h_2}$ is a solution, where α_1 and α_2 are real numbers and

$$\begin{aligned} x_p &= \begin{bmatrix} -2.2131153 & -0.6928850 & 0.8699073 & 0.21368862 \end{bmatrix}^\top \\ x_{h_1} &= \begin{bmatrix} 0.42222 & -0.336601 & 0.833975 & -0.113645 \end{bmatrix}^\top \\ x_{h_2} &= \begin{bmatrix} 0.16992 & -0.68904 & -0.275786 & 0.648301 \end{bmatrix}^\top \end{aligned}$$

Note that x_{h_1} and x_{h_2} span the nullspace of the matrix A. The particular solution x_p is the minimum norm solution out of all possible solutions and is given by

$$x_p = A^\top (AA^\top)^{-1} b,$$

where b is the right hand side of the equation above.

(c)

$$\begin{aligned} 13x_1 - 19x_2 &= 4 \\ -14x_1 + 10x_2 &= -14 \\ -7x_1 + 10x_2 &= 17 \\ 12x_1 - 7x_2 &= 82 \end{aligned}$$

Solution: This system of equations does not have a solution since the vector on the right hand side does not lie in the range space. The solution that minimizes the error $\|Ax - b\|$ is

$$x^* = \left(A^\top A\right)^{-1} A^\top b = \begin{bmatrix} 7.2016185 & 5.3706157 \end{bmatrix}^\top,$$

where b is the right-hand-side of the equation above.

3. Do the following computations

   (a) Compute the norm of the vectors $x = \begin{bmatrix} 18 & -11 & 7 \end{bmatrix}^\top$, $y = \begin{bmatrix} -12 & -5 & 20 \end{bmatrix}^\top$.

Solution: $\|x\| = 22.226$, $\|y\| = 23.8537$

(b) Compute the angle between the vectors $x = \begin{bmatrix} 16 & 10 & -10 \end{bmatrix}^\top$, $y = \begin{bmatrix} -17 & 9 & -12 \end{bmatrix}^\top$.

Solution: $\measuredangle(x, y) = 97.3577°$

(c) Is the following set of vectors independent? If not, why?

$$\begin{aligned} e_1 &= \begin{bmatrix} -5 & \frac{5}{9} & -\frac{35}{9} \end{bmatrix}^\top \\ e_2 &= \begin{bmatrix} \frac{25}{9} & \frac{5}{3} & -\frac{5}{9} \end{bmatrix}^\top \\ e_3 &= \begin{bmatrix} -\frac{185}{9} & 5 & -\frac{115}{9} \end{bmatrix}^\top \end{aligned}$$

Solution: No, they are not: $e_3 = 3e_1 + 2e_2$.

Explanation of Minimum Error Solution

Let us look at a system of equations $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ with $m > n$. Note that this implies $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. This is the set up of question $2(c)$.

As we discussed in class, in general, this means that $b \notin \text{range}(A)$. In other words, we cannot express b as a linear combination of the columns of A. The best we can hope to do is to minimize the error we are making when we are trying to satisfy the equation $Ax = b$. In other words, we are interested in the solution that minimizes $\|Ax - b\|^2$. Here are the steps to do this.

First, note that $\|v\|^2 = v^\top v$ for any $v \in \mathbb{R}^p$. Therefore, we can write

$$\begin{aligned} \|Ax - b\|^2 &= \left(x^\top A^\top - b^\top\right)(Ax - b) = x^\top A^\top Ax - x^\top A^\top b - b^\top Ax + b^\top b \\ &= x^\top A^\top Ax - 2x^\top Ab + b^\top b \end{aligned} \tag{1}$$

Now, we take the derivative of this expression with respect to x and set the result to zero. To do this, let us take the partial derivatives of the first two terms on the right-hand side of equation (1) individually:

$$\begin{aligned} \frac{\partial}{\partial x_l} x^\top A^\top Ax &= \frac{\partial}{\partial x_l} \sum_{i,j=1}^{n} \sum_{k=1}^{m} a_{ki} a_{kj} x_i x_j = \sum_{i,j=1}^{n} \sum_{k=1}^{m} a_{ki} a_{kj} \left(\delta_{il} x_j + x_i \delta_{jl}\right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} a_{kl} a_{kj} x_j + \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ki} a_{kl} x_i = 2 \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ki} a_{kj} x_j = 2A^\top Ax. \end{aligned}$$

where the first and last equalities follows from the definition of matrix multiplication, the second from product rule of differentiation, the third follows from the definition of the Kronecker delta, the fourth from relabeling the dummy summation index. Similarly, for the second term, we have

$$\frac{\partial}{\partial x_l} 2x^\top Ab = 2\frac{\partial}{\partial x_l} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i b_j = 2 \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \delta_{il} b_j = 2 \sum_{j=1}^{n} a_{jl} b_j = 2A^\top b.$$

where the first and last equalities follow from the definition of the matrix product and the inner equalities follow from rules of differentiation and the definition of Kronecker delta. Since the derivative of the term $b^\top b$ w.r.t x is zero, setting the derivative equal to zero gives

$$\frac{\partial}{\partial x} \|Ax - b\|^2 = 2\left(A^\top Ax - A^\top b\right) = 0$$

Solving this equation for x by inverting $A^\top A$, which is always possible as long as the columns of A are independent, we get the minimum norm solution as

$$\boxed{x = \left(A^\top A\right)^{-1} A^\top b}.$$

Explanation of the Minimum Norm Solution

Let us consider a system of equations $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ with $m < n$. Note that this implies $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. This is the set up of question 2(b).

In this case, there are infinitely many solutions to this set of equations. It is common to seek a solution x with the minimum norm. That is, we would like to solve the optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|x\|_2^2 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

To perform this minimization, we will use Lagrange multipliers. Define the Lagrangian

$$\mathcal{L}(x, \mu) = \|x\|_2^2 + \mu^\top (b - Ax)$$

We use the earlier rules of differentiation to take the derivative of the Lagrangian with respect to x and μ:

$$\frac{\partial}{\partial x}\mathcal{L} = 2x - A^\top \mu, \quad \frac{\partial}{\partial \mu}\mathcal{L} = b - Ax.$$

Setting these derivatives to zero, we get

$$x = \frac{1}{2} A^\top \mu, \quad Ax = b. \tag{2}$$

Plug the first equation into the second to get

$$b = \frac{1}{2} A A^\top \mu.$$

Assuming $AA^\top$ is invertible, which is the case if the matrix A is full rank, gives that $\mu = 2\left(AA^\top\right)^{-1} b$. Plugging this back into the first equation in (2) gives

$$\boxed{x = A^\top \left(AA^\top\right)^{-1} b}.$$