Ancillary Material #2 Solutions

(1) (a) (i) $3.5 < 3.25 + 1.5 + 1$, so it is a mechanism.

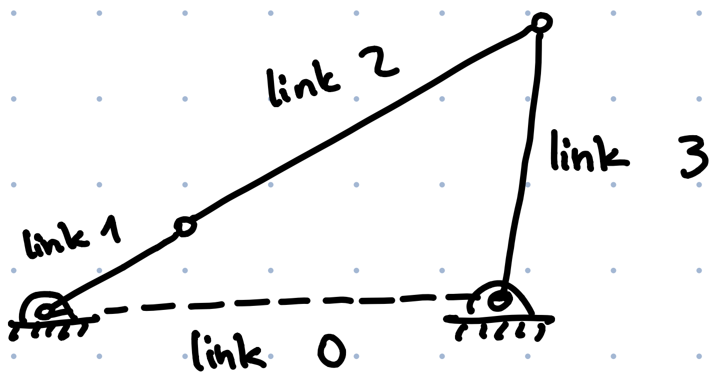
(ii) $2.5 < 2.5 + 1 + 2$, so it is a mechanism.

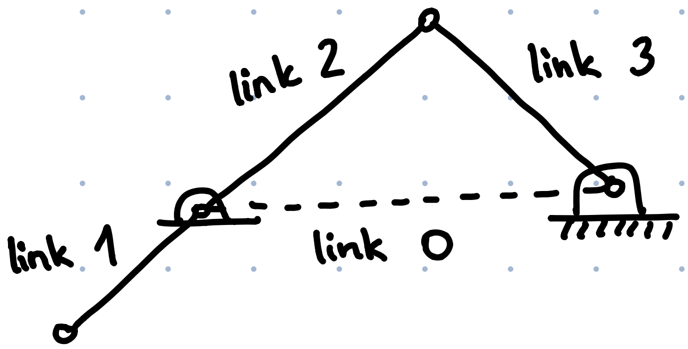
(b) (i) $l_{max} + l_{min} < l_a + l_b$ and since $l_0 = l_{min}$, it is a drag-link mechanism.

(ii) $l_{max} + l_{min} < l_a + l_b$, and since $l_1 = l_{min}$, it is a crank-rocker mechanism.

(c) (i) No limiting position.

(ii)





(2) (a) $$x_0 = x_1 + y_1 + 3z_1$$
$$y_0 = 2x_1 - y_1 + 4z_1$$
$$z_0 = 3x_1 + 5z_1$$

We want to find a matrix $T : \mathbb{R}^3 \to \mathbb{R}^3$ or $T \in \mathbb{R}^{3\times 3}$ such that whenever a vector v is expressed in Σ_0, Tv gives the expression of v in Σ_1, i.e.,

$$v^1 = Tv^0.$$

Let us consider the following vectors expressed in Σ_1:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}.$$

When these vectors are expressed in Σ_0, they are respectively e_1, e_2, e_3; the standard unit vectors, i.e.,

$$T^{-1}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \; T^{-1}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \; T^{-1}\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Written another way, these equations tell us,

$$T^{-1}\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} = I \Rightarrow T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

(b) T takes vectors expressed in Σ_0 to Σ_1. Hence T^{-1} takes vectors expressed in Σ_1 to Σ_0. We want to compute $T^{-1}w$. To do this, we need to compute T^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{array}\right] \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -3 & -1 & 1 & 0 \\ 0 & -2 & -4 & -3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & -1/3 & 0 \\ 0 & 1 & 2 & 3/2 & 0 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 7/6 & 1/3 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5/6 & -2/3 & 1/2 \\ 0 & 0 & 1 & 7/6 & 1/3 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -5/2 & -1 & 3/2 \\ 0 & 1 & 0 & -5/6 & -2/3 & 1/2 \\ 0 & 0 & 1 & 7/6 & 1/3 & -1/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/6 & 1/3 & 1/2 \end{array}\right.$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -5/6 & -2/3 & 1/2 \\ 0 & 0 & 1 & 7/6 & 1/3 & -1/2 \end{array}\right)$$

Therefore, $T^{-1} = \frac{1}{6}\begin{bmatrix} -5 & 2 & 3 \\ -5 & -4 & 3 \\ 7 & 2 & -3 \end{bmatrix}$

Hence

$$T^{-1}\omega = \frac{1}{6}\begin{bmatrix} -5 & 2 & 3 \\ -5 & -4 & 3 \\ 7 & 2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{6}\left(\begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix} + \begin{bmatrix} 4 \\ -8 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 9 \\ -9 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

(3) (a) $\dfrac{v}{\|v\|} = \dfrac{(18 \;\; -11 \;\; 7)^T}{\sqrt{324+121+49}} = \begin{pmatrix} 0.810 \\ -0.495 \\ 0.315 \end{pmatrix}$

(b) $\cos(\theta) = \dfrac{v \cdot w}{\|v\| \|w\|} = -0.12806 \Rightarrow \theta = 1.699$

$\approx 97.36^\circ$

(c) $v \times w = \begin{pmatrix} -5 \\ -5 \\ -1 \end{pmatrix}$

$\dfrac{v \times w}{\|v \times w\|} = \dfrac{(5 \;\; 5 \;\; 1)^T}{\sqrt{51}}$.

(d) Consider the matrix

$$A = \begin{bmatrix} -5 & 25/9 & -\frac{185}{9} \\ \frac{5}{9} & 5/3 & 5 \\ -\frac{35}{9} & -\frac{5}{9} & -\frac{115}{9} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -45 & 25 & -185 \\ 5 & 15 & 45 \\ -35 & -5 & -115 \end{bmatrix}$$

$$= \frac{5}{9} \begin{bmatrix} -9 & 5 & -37 \\ 1 & 3 & 9 \\ -7 & -1 & -23 \end{bmatrix}$$

$$\det(A) = \frac{125}{729} \left(-9 \cdot \begin{vmatrix} 3 & 9 \\ -1 & -23 \end{vmatrix} - \begin{vmatrix} 5 & -37 \\ -1 & -23 \end{vmatrix} - 7 \begin{vmatrix} 5 & -37 \\ 3 & 9 \end{vmatrix} \right)$$

$$= \frac{125}{729} (540 + 152 - 1092) \approx -68.59 \neq 0.$$

(4)

$$\langle f_i, f_j \rangle = \begin{matrix} 1 & \text{if} & i=j \\ 0 & \text{if} & i \neq j \end{matrix} = \delta^i_j .$$

Let us study the inner product on $\mathbb{R}[x]_2$.
An arbitrary element of $\mathbb{R}[x]_2$ has the form:

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 .$$

Thus, if $q(x) \in \mathbb{R}[x]_2$ s.t. $q(x) = \beta_0 + \beta_1 x + \beta_2 x^2$, then, we have

$$\langle p, q \rangle = \int_0^1 (\alpha_0 + \alpha_1 x + \alpha_2 x^2)(\beta_0 + \beta_1 x + \beta_2 x^2)\, dx$$

$$= \int_0^1 \Big\{ \alpha_0\beta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0)x + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0)x^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)x^3 + \alpha_2\beta_2 x^4 \Big\}\, dx$$

$$= \alpha_0\beta_0 + \frac{1}{2}(\alpha_0\beta_1 + \alpha_1\beta_0) + \frac{1}{3}(\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0) + \frac{1}{4}(\alpha_1\beta_2 + \alpha_2\beta_1) + \frac{1}{5}\alpha_2\beta_2 . \qquad (*)$$

Thus, if we represent the polynomials p, q in the basis $\{1, x, x^2\}$ for $\mathbb{R}[x]_2$ by $\bar{p}$ and $\bar{q}$, we can express the inner product $\langle p, q \rangle$ by

$\langle p, q \rangle = \bar{p}^T M \bar{q}$, where, M can be read off from $(*)$ as

$$M = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Now, let's apply the constraints for f_3. We have $\langle f_1, f_3\rangle = \langle f_2, f_3\rangle = 0$ and $\langle f_3, f_3\rangle = 1$.

Let $f_3(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. Then

$$0 = \langle f_1, f_3\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}^T M \begin{pmatrix}\alpha_0\\\alpha_1\\\alpha_2\end{pmatrix} = \alpha_0 + \frac{1}{2}\alpha_1 + \frac{1}{3}\alpha_2 .$$

$$0 = \langle f_2, f_3\rangle = \begin{pmatrix}\sqrt{3}\\-2\sqrt{3}\\0\end{pmatrix}^T M \begin{pmatrix}\alpha_0\\\alpha_1\\\alpha_2\end{pmatrix} = \sqrt{3}\alpha_0 + \frac{\sqrt{3}}{2}\alpha_1 + \frac{\sqrt{3}}{3}\alpha_2 - \sqrt{3}\alpha_0 - \frac{2\sqrt{3}}{3}\alpha_1 - \frac{\sqrt{3}}{2}\alpha_2$$

$$= -\frac{\sqrt{3}}{6}(\alpha_1 + \alpha_2)$$

$$1 = \langle f_3, f_3\rangle = \alpha_0^2 + \alpha_0\alpha_1 + \frac{1}{3}(2\alpha_0\alpha_2 + \alpha_1^2) + \frac{1}{2}\alpha_1\alpha_2 + \frac{1}{5}\alpha_2^2$$

Hence, we get the system of polynomial equations to solve for $\alpha_0, \alpha_1, \alpha_2$: $[\alpha_1 = -\alpha_2]$

$$\alpha_0 + \frac{1}{6}\alpha_1 = 0$$

$$\alpha_0^2 + \alpha_0\alpha_1 + \frac{1}{3}(-2\alpha_0\alpha_1 + \alpha_1^2) \underbrace{- \frac{1}{2}\alpha_1^2 + \frac{1}{5}\alpha_1^2}_{-3/10\,\alpha_1^2}$$

$$= \alpha_0^2 + \frac{1}{3}\alpha_0\alpha_1 + \frac{1}{30}\alpha_1^2 = 1.$$

Substituting from the top to the bottom:

$$1 = \alpha_0^2 - 2\alpha_0^2 + \frac{6}{5}\alpha_0^2 \Rightarrow \alpha_0 = \pm\sqrt{5}\,; \quad \alpha_1 = \mp 6\sqrt{5}\,; \quad \alpha_2 = \pm 6\sqrt{5}$$