ME 380
Kinematics and Machine Dynamics
Homework VI Solutions

1. [25 points] For the planar mechanism shown in Figure 1, draw the free body diagram for each link. Write down equations of motion for each link. Identify number of equations and all the unknowns. Put equations in a matrix form.

   Assume uniform links with center of mass located at the link centers. Gravity is out of plane. Links have masses m_i and inertias J_i, $i = 1, \ldots, 5$ about their centers of mass, in the direction perpendicular to the plane. Assume that kinematic analysis has already been performed and all accelerations are known. No friction exists at the joints. (Hint: To simplify free body diagrams at the multiple link junction, consider point J as an auxiliary rigid body with no mass.)

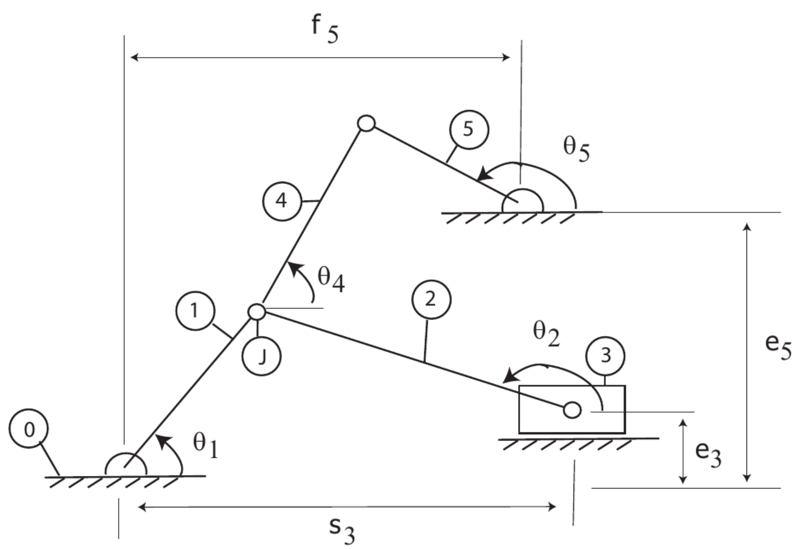
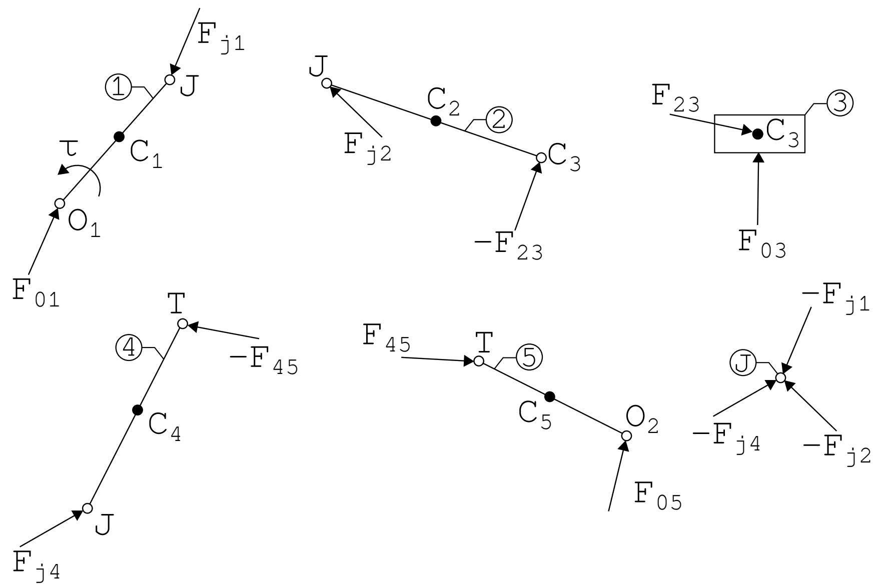


Figure 1: Planar Mechanism

Solution: Figure below shows the free body diagram of each link of this planar mechanism. The force and moment dynamic balance equations about the center of mass are then given by



$$F_{01} + F_{j1} - m_1\,{}^0a^{C_1} = 0, \tag{1a}$$

$$F_{j2} - F_{23} - m_2\,{}^0a^{C_2} = 0, \tag{1b}$$

$$F_{23} + F_{03} - m_3\,{}^0a^{C_3} = 0, \tag{1c}$$

$$F_{j4} - F_{45} - m_4\,{}^0a^{C_4} = 0, \tag{1d}$$

$$F_{45} + F_{05} - m_5\,{}^0a^{C_5} = 0, \tag{1e}$$

$$F_{j1} + F_{j2} + F_{j4} = 0, \tag{1f}$$

$$M^{1/C_1} = p^{C_1O_1} \times F_{01} + p^{C_1J} \times F_{j1} + \tau - J_1\,{}^0\alpha^1 = 0, \tag{1g}$$

$$M^{2/C_2} = p^{C_2J} \times F_{j2} - p^{C_2C_3} \times F_{23} - J_2\,{}^0\alpha^2 = 0, \tag{1h}$$

$$M^{4/C_4} = p^{C_4J} \times F_{j4} - p^{C_4T} \times F_{45} - J_4\,{}^0\alpha^4 = 0, \tag{1i}$$

$$M^{5/C_5} = p^{C_5T} \times F_{45} + p^{C_5O_2} \times F_{05} - J_5\,{}^0\alpha^5 = 0, \tag{1j}$$

There are 16 scalar equations and 16 unknowns, listed as

$$\xi = (F_{01}, F_{03}, F_{23}, F_{45}, F_{05}, F_{j1}, F_{j2}, F_{j4}, \tau)\,,$$

each of whose x- and y-components are unknown except for F_{03} and τ. The position vectors in equation (1) are given by

$$
\begin{aligned}
p^{C_1O_1} &= -\frac{l_1}{2}\cos(\theta_1)\hat{e}_1 + \sin(\theta_1)\hat{e}_2, \quad p^{C_1J} = \frac{l_1}{2}\cos(\theta_1)\hat{e}_1 + \sin(\theta_1)\hat{e}_2, \\
p^{C_2J} &= -\frac{l_2}{2}\cos(\theta_2)\hat{e}_1 + \sin(\theta_2)\hat{e}_2, \quad p^{C_2C_3} = \frac{l_2}{2}\cos(\theta_2)\hat{e}_1 + \sin(\theta_2)\hat{e}_2, \\
p^{C_4J} &= -\frac{l_4}{2}\cos(\theta_4)\hat{e}_1 + \sin(\theta_4)\hat{e}_2, \quad p^{C_4T} = \frac{l_4}{2}\cos(\theta_4)\hat{e}_1 + \sin(\theta_4)\hat{e}_2, \\
p^{C_5T} &= -\frac{l_5}{2}\cos(\theta_5)\hat{e}_1 + \sin(\theta_5)\hat{e}_2, \quad p^{C_5O_2} = \frac{l_5}{2}\cos(\theta_5)\hat{e}_1 + \sin(\theta_5)\hat{e}_2.
\end{aligned}
$$

Refer to Figure 2 for questions 2, 3, and 4. The link lengths are given by $l_0 = 0.25\,\mathrm{m}$, $l_1 = l_4 = 0.3\,\mathrm{m}$, $l_2 = l_3 = 0.35\,\mathrm{m}$. Each link is made of homogeneous material, so assume that the center of mass of each link is at its geometric center. The inertial constants are given by $m_1 = m_4 = 0.15\,\mathrm{kg}$, $m_2 = m_3 = 0.1\,\mathrm{kg}$ and $I_1 = I_4 = 1.5 \times 10^{-3}\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_2 = I_3 = 2 \times 10^{-3}\,\mathrm{kg}\cdot\mathrm{m}^2$. Gravity acts downwards within the plane.

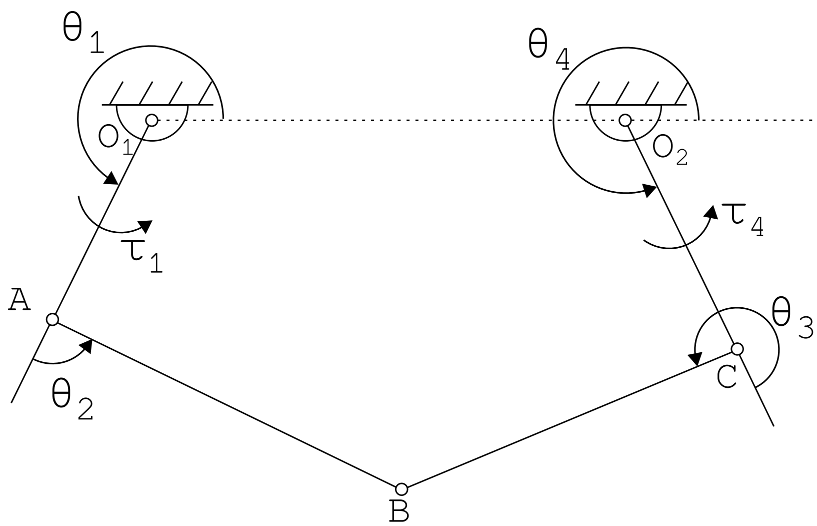
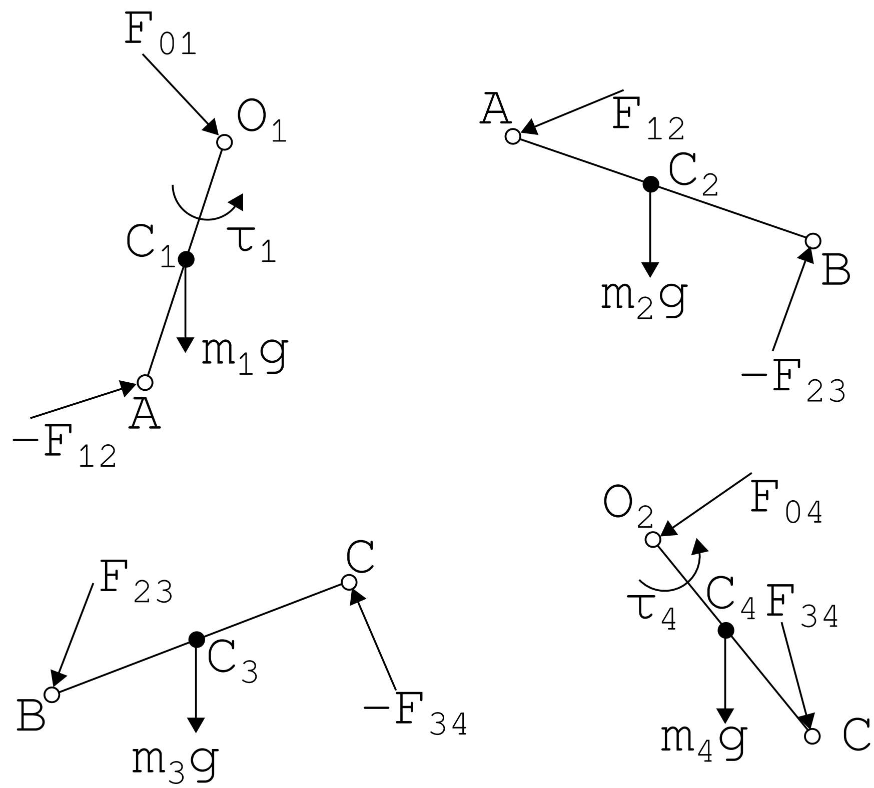


Figure 2: `Pantograph`

2. [25 points] Perform position-level kinematic analysis for the pantograph depicted in Figure 2. Follow this analysis up by performing a static-force analysis. Find the values of the motor torque $\tau = \begin{bmatrix}\tau_1 & \tau_4\end{bmatrix}^\top$ necessary to "cancel the gravity" (i.e. keep the mechanism stationary) when $\begin{bmatrix}\theta_1 & \theta_4\end{bmatrix}^\top = \begin{bmatrix}240^\circ & 310^\circ\end{bmatrix}^\top$. (Notice that you first need to find what θ_2 and θ_3 correspond to these values of θ_1 and θ_4.)

Solution: Solving the position-level forward kinematics with a numerical technique yields $\theta_2 = 90.890°$ and $\theta_3 = -95.098°$.

The figure below shows the free body diagrams of each link, excluding the inertial forces.



The Newton-Euler formulation for static equilibrium states that the sum of all the forces and the sum of the moments about any point on a rigid body must equal zero.

The sum of the forces R_i on rigid body i is given by

$$R_i = F_{i-1,i} - F_{i,i+1} - m_i g \hat{e}_2 = 0, \quad i \in \{1, 2, 3\}, \tag{2a}$$

$$R_4 = F_{34} + F_{04} - m_4 g \hat{e}_2 = 0. \tag{2b}$$

The sum of the moments about the center of mass of each link are given by

$$M^{1/C_1} = p^{C_1 O_1} \times F_{01} - p^{C_1 A} \times F_{12} + \tau_1 e_3 = 0, \tag{3a}$$

$$M^{2/C_2} = p^{C_2 A} \times F_{12} - p^{C_2 B} \times F_{23} = 0, \tag{3b}$$

$$M^{3/C_3} = p^{C_3 B} \times F_{23} - p^{C_3 C} \times F_{34} = 0, \tag{3c}$$

$$M^{4/C_4} = p^{C_4 C} \times F_{34} + p^{C_4 O_2} \times F_{04} + \tau_4 e_3 = 0, \tag{3d}$$

where the various position vectors are given in coordinates by

$$
\begin{aligned}
p^{C_1O_1} &= -p^{C_1A} = -\frac{l_1}{2}\begin{bmatrix}\cos(\theta_1) & \sin(\theta_1) & 0\end{bmatrix}^\top, \\
p^{C_2A} &= -p^{C_2B} = -\frac{l_2}{2}\begin{bmatrix}\cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0\end{bmatrix}^\top, \\
p^{C_3B} &= -p^{C_3C} = \frac{l_3}{2}\begin{bmatrix}\cos(\theta_3+\theta_4) & \sin(\theta_3+\theta_4) & 0\end{bmatrix}^\top, \\
p^{C_4C} &= -p^{C_4O_2} = \frac{l_4}{2}\begin{bmatrix}\cos(\theta_4) & \sin(\theta_4) & 0\end{bmatrix}^\top.
\end{aligned}
$$

Putting equations (2) and (3) into matrix form yields the system $Ax = b$ (3 significant digits are displayed) where $x = \begin{bmatrix}F_{01} & F_{12} & F_{23} & F_{34} & F_{04} & \tau\end{bmatrix}^\top$ and

$$
A = \begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-0.13 & 0.075 & -0.13 & 0.075 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -0.085 & -0.153 & -0.085 & -0.153 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.1 & -0.144 & 0.1 & -0.144 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.115 & 0.096 & -0.115 & -0.096 & 0 & 1
\end{bmatrix},
$$

$$
b = \begin{bmatrix}0 & 1.472 & 0 & 0.981 & 0.0 & 0.981 & 0 & 1.472 & 0 & 0 & 0 & 0\end{bmatrix}^\top.
$$

Solving this system of linear equations yields, in particular,

$$
\tau = \begin{bmatrix}-0.45242 \\ 0.52138\end{bmatrix}.
$$

3. [25 points] Complete the kinematic analysis of the pantograph by perfoming a velocity- and acceleration-level analysis. Given that $\begin{bmatrix}\theta_1 & \theta_4\end{bmatrix}^\top = \begin{bmatrix}240^\circ & 310^\circ\end{bmatrix}^\top$ and $\begin{bmatrix}\dot\theta_1 & \dot\theta_4\end{bmatrix}^\top = \begin{bmatrix}-15 & 5\end{bmatrix}^\top \frac{\text{rad}}{\text{s}}$, at an instant in time, find $\begin{bmatrix}\theta_2 & \theta_3\end{bmatrix}^\top$ and $\begin{bmatrix}\dot\theta_2 & \dot\theta_3\end{bmatrix}^\top$. Additionally, find the angular acceleration of all links at this instant if $\tau = \begin{bmatrix}\tau_1 & \tau_4\end{bmatrix}^\top = \begin{bmatrix}1 & -1\end{bmatrix}^\top$ Nm.

Solution: Solving the velocity-level kinematics with the given velocities of links 1 and 4 yields

$$\begin{bmatrix}\dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} = \begin{bmatrix}25.816\\ -21.002\end{bmatrix}\frac{\text{rad}}{\text{s}}.$$

We append the static-force balance equations the inertial forces and change our unknown from τ to $\ddot{\theta} = \begin{bmatrix}\ddot{\theta}_1 & \ddot{\theta}_2 & \ddot{\theta}_3 & \ddot{\theta}_4\end{bmatrix}^\top$. Thus equations (2) and (3) become

$$\begin{aligned}
R_i &= F_{i-1,i} - F_{i,i+1} - m_i g\hat{e}_2 - m_i{}^0a^{C_i} = 0, \quad i \in \{1,2,3\}, && \text{(4a)}\\
R_4 &= F_{34} + F_{04} - m_4 g\hat{e}_2 - m_4{}^0a^{C_4} = 0, && \text{(4b)}\\
M^{1/C_1} &= p^{C_1O_1} \times F_{01} - p^{C_1A} \times F_{12} + \tau_1 e_3 - I_1{}^0\alpha^1 = 0, && \text{(4c)}\\
M^{2/C_2} &= p^{C_2A} \times F_{12} - p^{C_2B} \times F_{23} - I_2{}^0\alpha^2 = 0, && \text{(4d)}\\
M^{3/C_3} &= p^{C_3B} \times F_{23} - p^{C_3C} \times F_{34} - I_3{}^0\alpha^3 = 0, && \text{(4e)}\\
M^{4/C_4} &= p^{C_4C} \times F_{34} + p^{C_4O_2} \times F_{04} + \tau_4 e_3 - I_4{}^0\alpha^4 = 0, && \text{(4f)}
\end{aligned}$$

where $\alpha_1 = \ddot{\theta}_1\hat{e}_3$, $\alpha_2 = (\ddot{\theta}_1+\ddot{\theta}_2)\hat{e}_3$, $\alpha_3 = (\ddot{\theta}_3+\ddot{\theta}_4)\hat{e}_3$, and $\alpha_4 = \ddot{\theta}_4\hat{e}_3$. In these equations, the accelerations of the centers of masses are given by

$$\begin{aligned}
{}^0a^{C_1} &= {}^0\alpha^1 \times p^{O_1C_1} + {}^0\omega^1 \times \left({}^0\omega^1 \times p^{O_1C_1}\right),\\
{}^0a^{C_2} &= {}^0\alpha^1 \times p_1 + {}^0\omega^1 \times \left({}^0\omega^1 \times p_1\right) + {}^0\alpha^2 \times p^{AC_2} + {}^0\omega^2 \times \left({}^0\omega^2 \times p^{AC_2}\right),\\
{}^0a^{C_3} &= -{}^0\alpha^4 \times p_4 - {}^0\omega^4 \times \left({}^0\omega^4 \times p_4\right) + {}^0\alpha^3 \times p^{CC_3} + {}^0\omega^3 \times \left({}^0\omega^3 \times p^{CC_3}\right),\\
{}^0a^{C_4} &= {}^0\alpha^4 \times p^{O_2C_4} + {}^0\omega^4 \times \left({}^0\omega^4 \times p^{O_2C_4}\right),
\end{aligned}$$

where $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AB}$, $p_3 = \overrightarrow{BC}$, $p_4 = \overrightarrow{CO_2}$. We need two more equations to be able to solve for all the unknowns. These come from the acceleration-level kinematic analysis as

$$\sum_{k=1}^{4} {}^0\alpha^k \times p_k + {}^0\omega^k \times \left({}^0\omega^k \times p_k\right) = 0. \tag{5}$$

After appending equations (4) and (5), we put them into matrix form $A_{dyn}x = b_{dyn}$ and solve for the unknowns $x = \begin{bmatrix}F_{01} & F_{12} & F_{23} & F_{34} & F_{04} & \ddot{\theta}\end{bmatrix}$. Let us decompose $A_{dyn} \in \mathbb{R}^{14\times 14}$ as follows

$$A_{dyn} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{12\times 10}$, $A_2 \in \mathbb{R}^{12\times 4}$, $0 \in \mathbb{R}^{2\times 10}$, and $A_3 \in \mathbb{R}^{2\times 4}$. Here, $A_1 = A \begin{bmatrix} I_{10} \\ 0 \end{bmatrix}$ where I and 0 are the identity and zero matrices of appropriate sizes and A is taken from problem 2. The remaining matrices A_2 and A_3 are given by (up to 4 digits)

$$A_2 = \begin{bmatrix} -0.0195 & 0 & 0 & 0 \\ 0.0112 & 0 & 0 & 0 \\ -0.0345 & -0.0085 & 0 & 0 \\ -0.0003 & -0.0153 & 0 & 0 \\ 0 & 0 & -0.01 & -0.033 \\ 0 & 0 & 0.0144 & -0.0049 \\ 0 & 0 & 0 & -0.0172 \\ 0 & 0 & 0 & -0.0145 \\ -0.0015 & 0 & 0 & 0 \\ -0.002 & -0.002 & 0 & 0 \\ 0 & 0 & -0.002 & -0.002 \\ 0 & 0 & 0 & -0.0015 \\ 0.4301 & 0.1703 & -0.2003 & -0.4301 \\ 0.1558 & 0.3058 & 0.287 & 0.0942 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.43 & 0.17 & -0.2 & -0.43 \\ 0.156 & 0.306 & 0.287 & 0.094 \end{bmatrix}.$$

The vector $b_{dyn} \in \mathbb{R}^{14}$ turns out to be (up to 3 digits)

$$b_{dyn} = \begin{bmatrix} 2.531 & 5.856 & 1.586 & 7.823 & 3.193 & 4.119 & -0.3616 & 1.902 & -1 & 0 & 0 & 1 & 70.702 & -21.351 \end{bmatrix}^\top$$

Solving $A_{dyn}x = b_{dyn}$ and recording the last 4 elements, which correspond to the second time derivatives of the joint angles, we find

$$\begin{bmatrix} \ddot{\theta}_1 & \ddot{\theta}_2 & \ddot{\theta}_3 & \ddot{\theta}_4 \end{bmatrix}^\top = \begin{bmatrix} 160.843 & -304.665 & 240.37 & -236.095 \end{bmatrix}^\top \frac{\text{rad}}{\text{s}^2}.$$

4. [25 points (bonus)] (Strongly recommended as this will help with your project immensely)

Assume that the motor torques equal identically to $\tau = \begin{bmatrix} -0.45 & 0.45 \end{bmatrix}^\top$. Suppose at the initial time ($t = 0$) the mechanism's state is $\begin{bmatrix} \theta_1 & \theta_4 & \dot{\theta}_1 & \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} 240^\circ & 310^\circ & 0 & 0 \end{bmatrix}$. Write a Julia or Matlab program that simulates the system for 20 seconds (finds θ_1, θ_4, $\dot{\theta}_1$, and $\dot{\theta}_4$ for all time $t \in [0, 20]$ using Newton-Euler equations). You may use the Julia package `DifferentialEquations`, Matlab's `ode45` function or simply implement a backward Euler integration scheme for numerical integration. Present the simulation results by plotting θ_1, θ_4, $\dot{\theta}_1$, and $\dot{\theta}_4$ as a function of time t.

Solution: The simulation results in the following evolution of system states