ME 380
Kinematics and Machine Dynamics
Homework V Solutions

1. [25 points] Perform position, velocity, and acceleration analysis for the mechanism depicted in Figure 1 using analytical vector methods. Assume $\omega_1 = 5\ \frac{\text{rad}}{\text{s}}$ and $\alpha_1 = 2\ \frac{\text{rad}}{\text{s}^2}$.

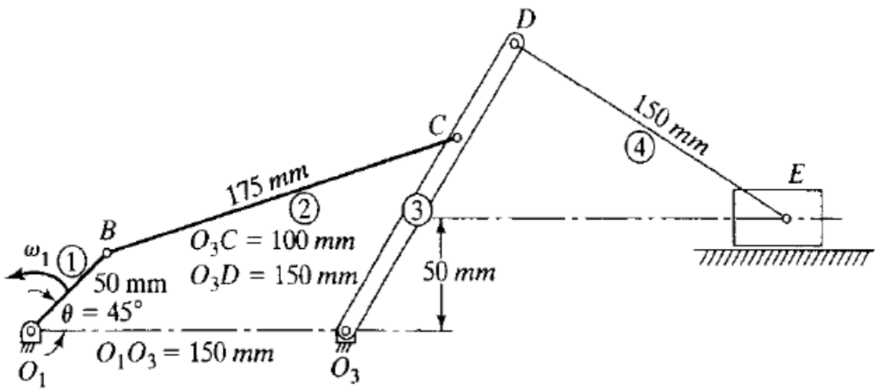

Figure 1: `Fourbar with slider`

Solution: Define the following quantities

$$p_0 = \overrightarrow{O_3O_1},\ p_1 = \overrightarrow{O_1B},\ p_2 = \overrightarrow{BC},\ p_3 = \overrightarrow{CO_3},\ p_4 = \overrightarrow{O_3D},\ p_5 = \overrightarrow{DE},\ p_6 = \overrightarrow{O_3E}.$$

Two loop equations we can write down are

$$\begin{aligned} p_0 + p_1 + p_2 + p_3 &= 0, \\ p_4 + p_5 - p_6 &= 0. \end{aligned} \tag{1}$$

Depending on the definition of the angles θ_2, θ_3, and θ_4, the expressions of these vector loop equations in the earth frame may take on different forms. The following uses absolute angles for which we have the relations $x_i = c_i x_0 + s_i y_0$ and $y_i = -s_i x_0 + c_i y_0$, for $i = 1, \ldots, 4$. Further, $x_5 = x_0$ and $y_5 = y_0$. In the frame Σ_0, equations (1) become

$$\begin{aligned} \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_3 \cos\theta_3 - l_0 \\ l_1 \sin\theta_1 + l_2 \sin\theta_2 - l_3 \sin\theta_3 \end{bmatrix} &= 0, \\ \begin{bmatrix} l_3' \cos\theta_3 - l_4 \cos\theta_4 - d_5 \\ l_3' \sin\theta_3 - l_4 \sin\theta_4 - a_0 \end{bmatrix} &= 0. \end{aligned} \tag{2}$$

where $\{l_i\}_0^4$ are the lengths of the vectors p_0, p_1, p_2, p_3 and p_5, l_3' is the length of p_4 and a_0 is the vertical component of the vector p_6.

Solving these equations, we get

θ_2	θ_3	θ_4	d_5
16.353°	57.808°	149.141°	208.68

Differentiating equations (1) w.r.t. time in frame 0, we get

$$\sum_{i=1}^{3} {}^0\omega^i \times p_i = 0,$$

(3)
$$\sum_{i=4}^{5} {}^0\omega^i \times p_i - \dot{d}_5 x_0 = 0.$$

Writing these equations in coordinates and putting into matrix form, we get $Jv = 0$, where $v = \begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 & \dot{d}_5\end{bmatrix}^\top$ and

$$J = \begin{bmatrix} -l_1\sin\theta_1 & -l_2\sin\theta_2 & l_3\sin\theta_3 & 0 & 0 \\ l_1\cos\theta_1 & l_2\cos\theta_2 & -l_3\cos\theta_3 & 0 & 0 \\ 0 & 0 & -l_3'\sin\theta_3 & l_4\sin\theta_4 & -1 \\ 0 & 0 & l_3'\cos\theta_3 & -l_4\cos\theta_4 & 0 \end{bmatrix}$$

Solving $Jv = 0$ at the current configuration for the given $\dot{\theta}_1 = \omega_1 = 5$, we get

$\dot{\theta}_2$	$\dot{\theta}_3$	$\dot{\theta}_4$	$\dot{d}_5$
−0.478	1.810	−1.12	−316.26

Differentiating equations (3) w.r.t. time in frame 0, we get

$$\sum_{i=1}^{3} {}^0\alpha^i \times p_i + {}^0\omega^i \times \left({}^0\omega^i \times p_i\right) = \sum_{i=1}^{2}\left(l_i\ddot{\theta}_i y_i - l_i\dot{\theta}_i^2 x_i\right) - l_3\ddot{\theta}_3 y_3 + l_3\dot{\theta}_3^2 x_3 = 0,$$

$$\sum_{i=4}^{5} {}^0\alpha^i \times p_i + {}^0\omega^i \times \left({}^0\omega^i \times p_i\right) - \ddot{d}_5 x_0 = l_3'\ddot{\theta}_3 y_3 - l_3'\dot{\theta}_3^2 x_3 - l_4\ddot{\theta}_4 y_4 + l_4\dot{\theta}_4^2 x_4 - \ddot{d}_5 x_0 = 0.$$

Writing these equations in frame Σ_0, we get $J\dot{v} + \dot{J}v = 0$, where J is as given above and $\dot{J}$ denotes the matrix whose components are the time derivatives of the components of J. Solving this equation for $\dot{v}$ gives the desired accelerations.

$\ddot{\theta}_2$	$\ddot{\theta}_3$	$\ddot{\theta}_4$	$\ddot{d}_5$
7.760	14.189	−6.329	−2712.519

2. [25 points] Perform position and velocity analysis for the mechanism depicted in Figure 2 using analytical vector methods. Assume $\omega_1 = 20\ \frac{\text{rad}}{\text{s}}$ (the only nonzero component of the angular velocity of the cylinder with respect to the fixed frame).

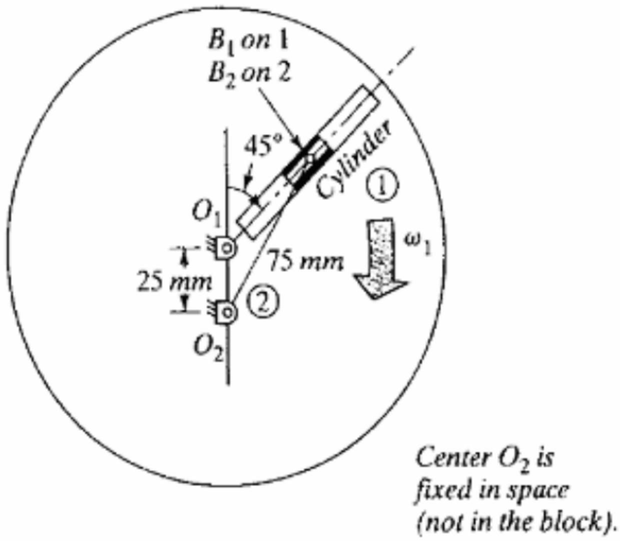

Figure 2

Solution: Let $p_0 = \overrightarrow{O_1O_2}$, $p_1 = \overrightarrow{O_1B_1}$, $p_2 = \overrightarrow{O_2B_2}$, and $r = \overrightarrow{B_1B_2}$. The position-level loop equation is then

$$p_0 - p_1 + p_2 - r = 0. \tag{4}$$

We place the coordinate frames by choosing z-axes to point perpendicularly out of the plane and by the following rules

- o_0 is at O_1 and x_0 points horizontally in the plane,
- o_1 is at B_1, x_1 points along p_1, and $\theta_1 = \measuredangle(x_0, x_1) = 90° - 45° = 45°$,
- o_2 is at B_2, x_2 points along p_2, and $\theta_2 = \measuredangle(-y_0, x_2)$,
- o_3 is at B, x_3 points along x_1, and $d_3 = |O_1B|$.

Writing equation (4) in coordinates, we get

$$l_0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + l_2 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} - d_3 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = 0.$$

Solving these equations for the unknowns yields $\theta_2 = 146.63°$ and $d_3 = 55.209$ [mm]. Differentiating equations (4) w.r.t. time in frame 0 yields

$$-{}^0\omega^1 \times p_1 + {}^0\omega^2 \times p_2 - {}^1v^{B_2} - {}^0\omega^1 \times r = 0.$$

Expressing this equation in coordinates, we get

$$-d_3\dot{\theta}_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} + l_2\dot{\theta}_2 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} - \dot{d}_3 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = 0.$$

solving these equations for the unknowns yields $\dot{\theta}_2 = -15.149 \left[\frac{\text{rad}}{\text{s}}\right]$, $\dot{d}_3 = 267.80$ $\left[\frac{\text{mm}}{\text{s}}\right]$.

3. [25 points] Perform position, velocity, and acceleration analysis for the mechanism shown in Figure 3 using analytical vector methods. Take $\theta_1 = 20°$, $\omega_1 = \frac{1}{2}\ \frac{\text{rad}}{\text{s}}$ and $\alpha_1 = 1\ \frac{\text{rad}}{\text{s}^2}$.

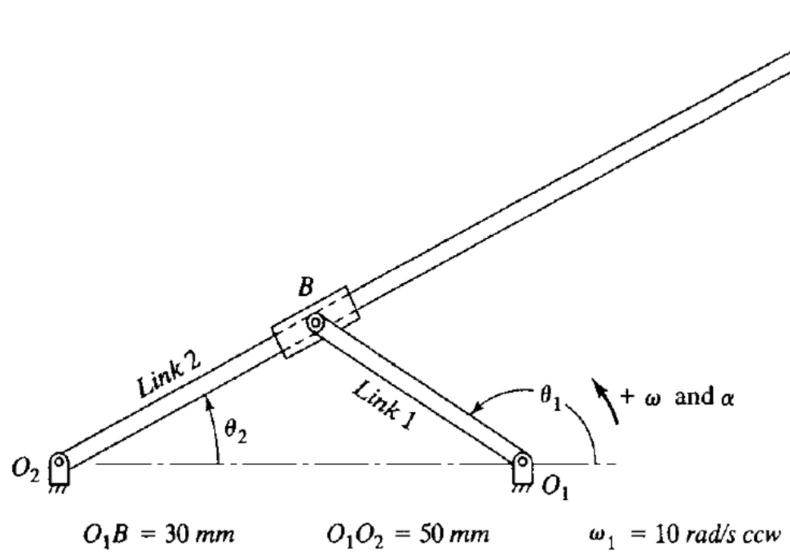

Figure 3

Solution: Let $p_0 = \overrightarrow{O_2O_1}$, $p_1 = \overrightarrow{O_1B_1}$, $p_2 = \overrightarrow{O_2B_2}$, and $r = \overrightarrow{B_2B_1}$, where B_1 and B_2 are points coincident with B at this instant. The position-level loop equation is then

$$p_0 + p_1 - r - p_2 = 0. \tag{5}$$

We place the coordinate frames by choosing z-axes to point perpendicularly out of the plane and by the following rules

- o_0 is at O_1 and x_0 points horizontally in the plane,
- o_1 is at B_1, x_1 points along p_1, and $\theta_1 = \measuredangle(x_0, x_1) = 20°$,
- o_2 is at B_2, x_2 points along p_2, and $\theta_2 = \measuredangle(x_0, x_2)$,
- o_3 is at B, x_3 points along x_2, and $d_3 = |O_2B|$.

Writing equation (4) in coordinates, we get

$$l_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} - d_3 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = 0.$$

Solving these equations for the unknowns yields $\theta_2 = 7.476°$ and $d_3 = 78.861$ [mm].

Differentiating equations (5) w.r.t. time in frame 0 yields

$${}^0\omega^1 \times p_1 - {}^2v^{B_1} - {}^0\omega^2 \times r - {}^0\omega^2 \times p_2 = 0.$$

Expressing this equation in coordinates, we get

$$l_1\dot{\theta}_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} + d_3\dot{\theta}_2 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} - \dot{d}_3 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = 0. \tag{6}$$

Solving these equations for the unknowns yields $\dot{\theta}_2 = 0.186 \ \left[\frac{\text{rad}}{\text{s}}\right]$, $\dot{d}_3 = -3.253 \ \left[\frac{\text{mm}}{\text{s}}\right]$.

Differentiating equations (6) w.r.t. time in frame 0 yields

$${}^0\alpha^1 \times p_1 + {}^0\omega^1 \times \left({}^0\omega^1 \times p_1\right) - {}^2a^{B_1} - 2{}^0\omega^2 \times {}^2v^{B_1} - {}^0\alpha^2 \times p_2 - {}^0\omega^2 \times \left({}^0\omega^2 \times p_2\right) = 0.$$

Expressing this equation in coordinates, we get

$$l_1\ddot{\theta}_1 \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix} + d_3\ddot{\theta}_2 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} - \ddot{d}_3 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = l_1\dot{\theta}_1^2 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} - 2\dot{\theta}_2\dot{d}_3 \begin{bmatrix} s_2 \\ -c_2 \end{bmatrix} - d_3\dot{\theta}_2^2 \begin{bmatrix} c_2 \\ s_2 \end{bmatrix}.$$

Solving these equations for the unknowns yields $\ddot{\theta}_2 = 0.366 \ \left[\frac{\text{rad}}{\text{s}^2}\right]$, $\ddot{d}_3 = -11.108 \ \left[\frac{\text{mm}}{\text{s}^2}\right]$.

4. [25 points] Derive forward and inverse kinematic equations (at position- and velocity-level) for the pantograph mechanism presented in Figure 4, using analytical vector methods. Feel free to define any variables required for the solution. Make sure to clearly mark your definitions on the figure and be consistent with your notation.

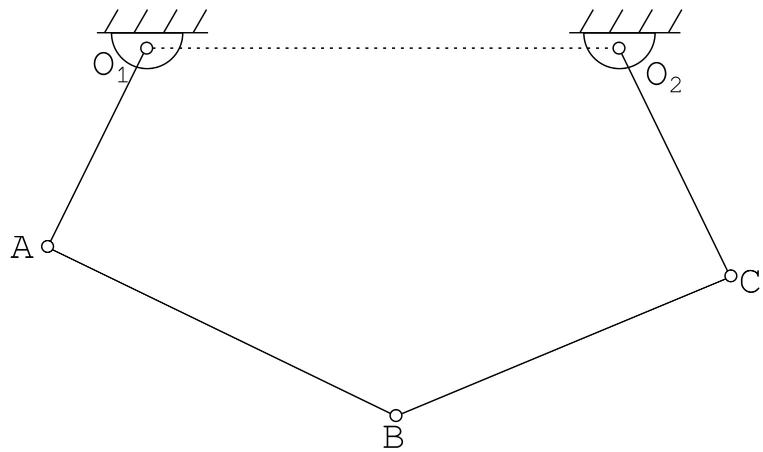

Figure 4: `Pantograph`

Solution: Let the origin of the inertial frame be placed at O_2 and define $p_0 = \overrightarrow{O_2O_1}$, $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AB}$, $p_3 = \overrightarrow{CB}$, and $p_4 = \overrightarrow{O_2C}$. Attach a coordinate frame Σ_k to the k^{th} rigid body, $k = 0, 1, \ldots 4$. Notice that the position vector p_k is fixed in frame Σ_k.

The position-level loop equation is given by

$$\sum_{i=0}^{2} p_i - \sum_{i=3}^{4} p_i = 0.$$

Differentiating this equation w.r.t. time in frame 0, we find, using differentiation in two frames formula that the velocity-level loop equation is given by

$$\sum_{i=1}^{2} {}^0\omega^i \times p_i - \sum_{i=3}^{4} {}^0\omega^i \times p_i = 0.$$

To find the position $\begin{bmatrix} x & y \end{bmatrix}^\top$ and the velocity $\begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}^\top$ of the point B in terms of the joint angles and their rates of change, we can use the relationships

$$\begin{aligned} \xi &:= \begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=0}^{2} p_i = \sum_{i=3}^{4} p_i, \\ \dot{\xi} &= \sum_{i=1}^{2} {}^0\omega^i \times p_i = \sum_{i=3}^{4} {}^0\omega^i \times p_i. \end{aligned}$$