ME 380
Kinematics and Machine Dynamics
Homework IV Solutions

1. [15 points] Consider the double-slider two-loop linkage illustrated in Figure 1, where $l_1 = 1.3$, $l_2 = 5.2$, $l_F = 5$, $l_{DE} = 4.5$, $l_{CD} = 1.5$, and $\theta_1 = \frac{\pi}{6}$. Find θ_2, θ_3, l_D, and l_E in two steps.

    1. Analytically derive the loop equations,
    2. Use numerical techniques to get a numerical solution.

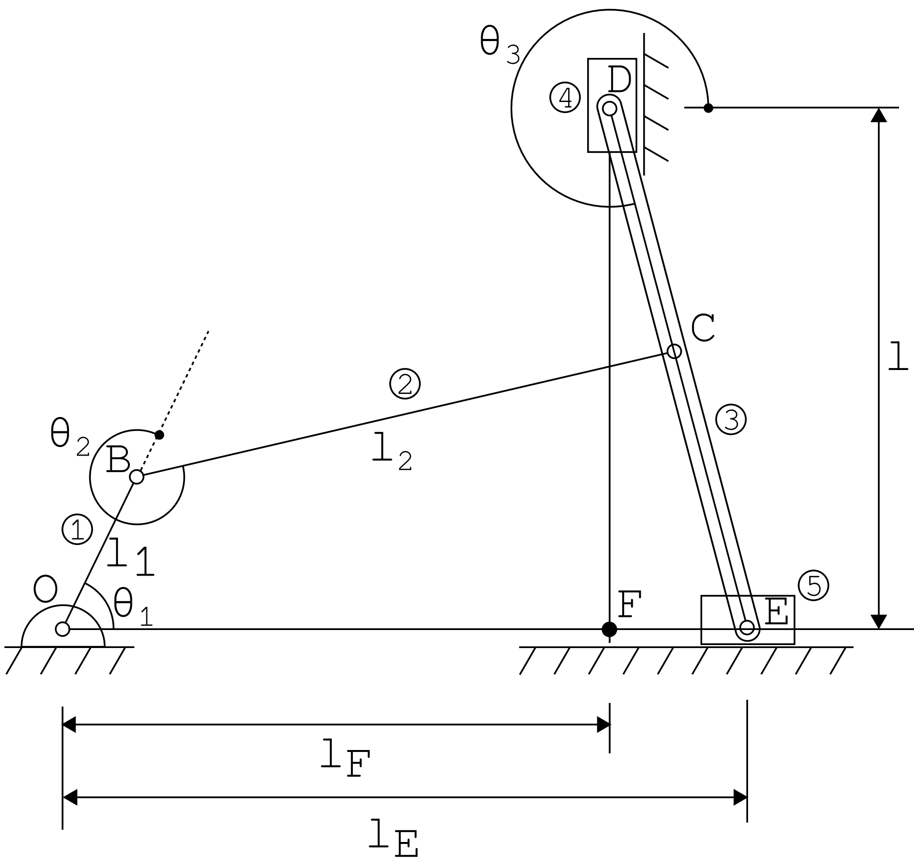


Figure 1: `Double-slider two-loop linkage`

Solution: Define the following quantities

$$p_0 = \overrightarrow{EO}, \quad p_1 = \overrightarrow{OB}, \quad p_2 = \overrightarrow{BC}, \quad p_3 = \overrightarrow{CE}, \quad p_4 = \overrightarrow{OD}, \quad p_5 = \overrightarrow{DE}.$$

In terms of these quantities, the two loop equations can be written as

$$p_0 + p_1 + p_2 + p_3 = 0, \tag{1a}$$
$$p_4 + p_5 + p_0 = 0. \tag{1b}$$

Writing these out in terms of the respective frames planted on links 1, 2, and 3, with the first axis aligned along the centerlines of the links, gives

$$-l_E\hat{e}_1 + l_1\hat{a}_1 + l_2\hat{b}_1 + l_3\hat{c}_1 = 0,$$
$$l_F\hat{e}_1 + l_D\hat{e}_2 + l_{DE}\hat{c}_1 - l_E\hat{e}_1 = 0.$$

where $l_3 = l_{DE} - l_{CD}$.

Expressing the various unit vectors along $\hat{e}_1$ and $\hat{e}_2$ yields

$$\begin{aligned}&[-l_E + l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_3)]\,\hat{e}_1\\ &\quad + [l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_3)]\,\hat{e}_2 = 0,\\ &[l_F - l_E + l_{DE}\cos(\theta_3)]\,\hat{e}_1 + [l_D + l_{DE}\sin(\theta_3)]\,\hat{e}_2 = 0.\end{aligned}$$

We have four equations in four unknowns. Dumping this into a Newton-Raphson method yields

θ_2	θ_3	l_D	l_E
$-17.002°$	$-37.339°$	2.729	8.578

2. [15 points] Referring to Figure 2, let $l_1 = 35$, $l_0 = 50$, $\omega_1 = 7.5\ \frac{rad}{s}$. and $\theta_1 = \frac{\pi}{10}$. Find
   a) l_2
   b) θ_2
   c) ω_2

Solution: Define the quantities

$$p_0 = \overrightarrow{O_2O_1}, \quad p_1 = \overrightarrow{O_1B_1}, \quad p_2 = \overrightarrow{O_2B_2}, \quad p_3 = \overrightarrow{B_2B_1}.$$

In terms of these quantities, the loop equation can be written as

$$p_2 + p_3 - p_1 - p_0 = 0. \tag{2}$$

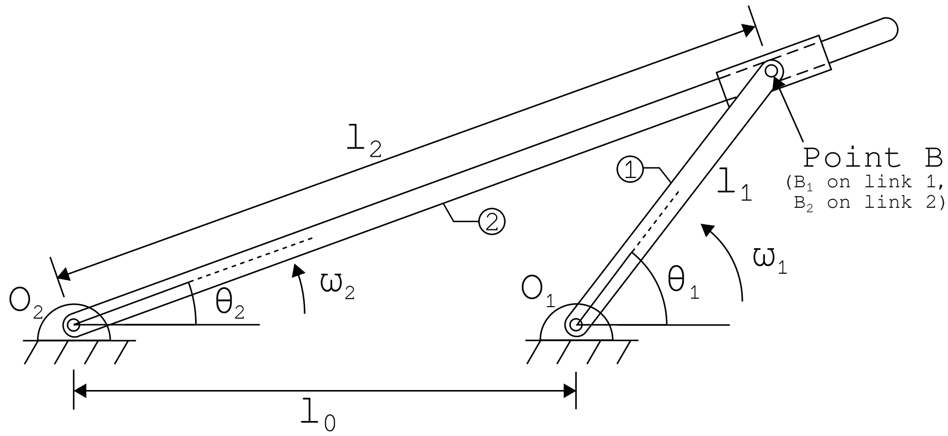


Figure 2: `Sliding contact linkage`

Differentiating this equation in the 0th frame yields

$$\begin{aligned} {}^{0}\omega^{2} \times p_2 + {}^{0}\frac{\mathrm{d}}{\mathrm{d}t}p_3 - {}^{0}\omega^{1} \times p_1 &= 0, \\ {}^{0}\omega^{2} \times p_2 + {}^{2}\frac{\mathrm{d}}{\mathrm{d}t}p_3 + {}^{0}\omega^{2} \times p_3 - {}^{0}\omega^{1} \times p_1 &= 0, \\ {}^{0}\omega^{2} \times p_2 + {}^{2}v^{B_1} - {}^{0}\omega^{1} \times p_1 &= 0. \end{aligned} \tag{3}$$

Writing out the position-level loop equation (2) in terms of various reference frames, we have

$$l_2\hat{b}_1 - l_1\hat{a}_1 - l_0\hat{e}_1 = 0, \tag{4a}$$

$$\left(-l_0 + l_2\cos(\theta_2) - l_1\cos(\theta_1)\right)\hat{e}_1 + \left(l_2\sin(\theta_2) - l_1\sin(\theta_1)\hat{e}_2\right) = 0. \tag{4b}$$

Solving these equations along the $\{\hat{e}_1, \hat{e}_2\}$ axes simultaneously yields $l_2 = 83.986$ and $\theta_2 = 7.399°$.

Now passing to the velocity-level loop equation (3) and writing it out in various reference frames, we have

$$\omega_2\hat{k} \times l_2\left(\cos(\theta_2)\hat{e}_1 + \sin(\theta_2)\hat{e}_2\right) + \dot{l}_2\hat{b}_1 - \omega_1\hat{k} \times l_1\left(\cos(\theta_1)\hat{e}_1 + \sin(\theta_1)\hat{e}_2\right) = 0.$$

where the unknows are $\dot{l}_2$ and ω_2. Writing out this equation in the world reference frame $\{\hat{e}_1, \hat{e}_2\}$ gives an equation of the form $Jx = f$,

$$\begin{bmatrix} \cos\theta_2 & -l_2\sin\theta_2 \\ \sin\theta_2 & l_2\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot{l}_2 \\ \omega_2 \end{bmatrix} = l_1 \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} \omega_1 \tag{5}$$

Plugging in the known values of θ_1, θ_2, l_1, and l_2 and solving yields $\dot{l}_2 = -48.292\frac{mm}{s}$ and $\omega_2 = 3.072\ \frac{rad}{s}$.

Analytic solution As an aside, we can solve equation (4b) analytically by first taking the ratio of these two equations to get

$$\tan(\theta_2) = \frac{l_1\sin\theta_1}{l_0 + l_1\cos\theta_1}.$$

Plugging this back in one of the equations in (4b) and noting that $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ (and $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$) gives

$$l_2 = \sqrt{l_0^2 + 2l_0l_1\cos(\theta_1) + l_1^2}.$$

It is much easier to solve equation (5) for $\dot{l}_2$ and ω_2 as it is linear. The inverse of the 2×2 matrix J, on the left hand side is given by

$$J^{-1} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\frac{1}{l_2}\sin\theta_2 & \frac{1}{l_2}\cos\theta_2 \end{bmatrix}.$$

One then readily computes

$$J^{-1}f = l_1\omega_1 \begin{bmatrix} \sin(\theta_2-\theta_1) \\ \frac{1}{l_2}\cos(\theta_1-\theta_2) \end{bmatrix}.$$

3. [15 points] Referring to Figure 3, let $l_0 = 30$, $l_1 = 15$, $l_2 = 35$, $l_3 = 20$, and $\theta_1 = \frac{\pi}{3}$. Calculate θ_2, θ_3, v_C, ω_2, and ω_3 for $\omega_1 = 20\ \frac{rad}{s}$.

Solution: Using case 3 of the vector cross product method, one readily obtains

$$p_2 = \overrightarrow{BC} = \begin{bmatrix} 34.895 & 2.705 \end{bmatrix}^\top, \quad p_3 = \overrightarrow{CO_3} = \begin{bmatrix} -12.395 & -15.696 \end{bmatrix}^\top.$$

Correspondingly,

$$\begin{aligned}\theta_2 &= \text{atan}_2\left(2.705, 34.895\right) = 0.077 \text{ rad} = 4.433^\circ \\ \theta_3 &= \text{atan}_2\left(-15.696, -12.395\right) = -2.239 \text{ rad} = -128.30^\circ.\end{aligned}$$

Differentiating the position-level vector loop equation $p_0 + p_1 + p_2 + p_3 = 0$ we get the velocity-level loop equation

$$^0\omega^1 \times p_1 + {}^0\omega^2 \times p_2 + {}^0\omega^3 \times p_3 = 0.$$

Writing out this equation along the inertial reference frame $\{\hat{e}_1, \hat{e}_2\}$ yields

$$\begin{bmatrix} -l_2 \sin\theta_2 & -l_3 \sin\theta_3 \\ l_2 \cos\theta_2 & l_3 \cos\theta_3 \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_3 \end{bmatrix} = l_1 \begin{bmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{bmatrix} \omega_1$$

Plugging in the knowns, $\{l_i, \theta_i\}_{i=1}^3$ and ω_1 and solving this linear equation yields $\omega_2 = 1.684 \frac{rad}{s}$ and $\omega_3 = 16.843 \frac{rad}{s}$.

Next, the position vector from the origin of the inertial frame to point C is given by $p = p_1 + p_2$. Differentiating this expression in this frame gives

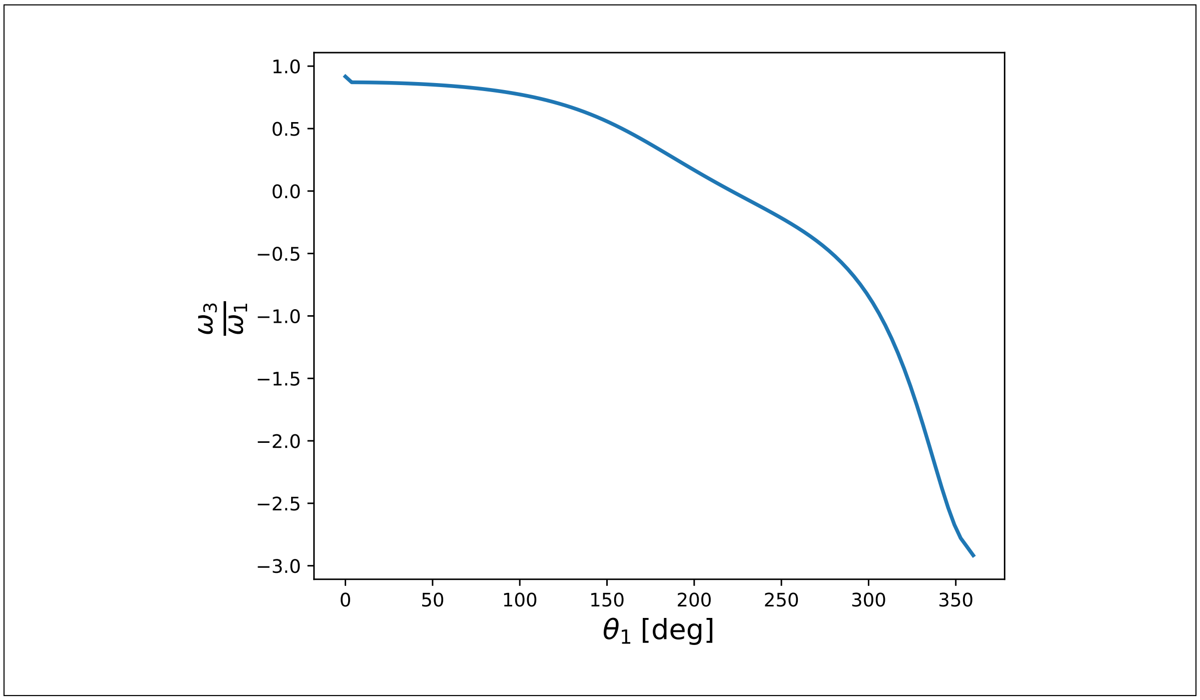
$$^0v^C = \frac{^0\mathrm{d}p}{\mathrm{d}t} = {}^0\omega^1 \times p_1 + {}^0\omega^2 \times p_2$$

We know everything on the right-hand side of this expression. Plugging in the relevant values gives $^0v^C = \begin{bmatrix} -264.36 & 208.77 \end{bmatrix}^\top \frac{mm}{s}$ and $\|^0v^C\| = 336.86 \frac{mm}{s}$.

4. [15 points] Plot $\frac{\omega_3}{\omega_1}$ versus θ_1 curve for the linkage depicted in Figure 3 if $\frac{l_0}{l_1} = 2$, $\frac{l_2}{l_1} = \frac{7}{3}$, and $\frac{l_3}{l_1} = \frac{4}{3}$.

Solution: If we knew θ_2 and θ_3, we could use equation (3) to compute $\frac{\omega_3}{\omega_1}$.

Given θ_1, we can compute θ_2 and θ_3 through vector cross product method programmatically. Using this, yields the following graph of $\frac{\omega_3}{\omega_1}$ versus θ_1.



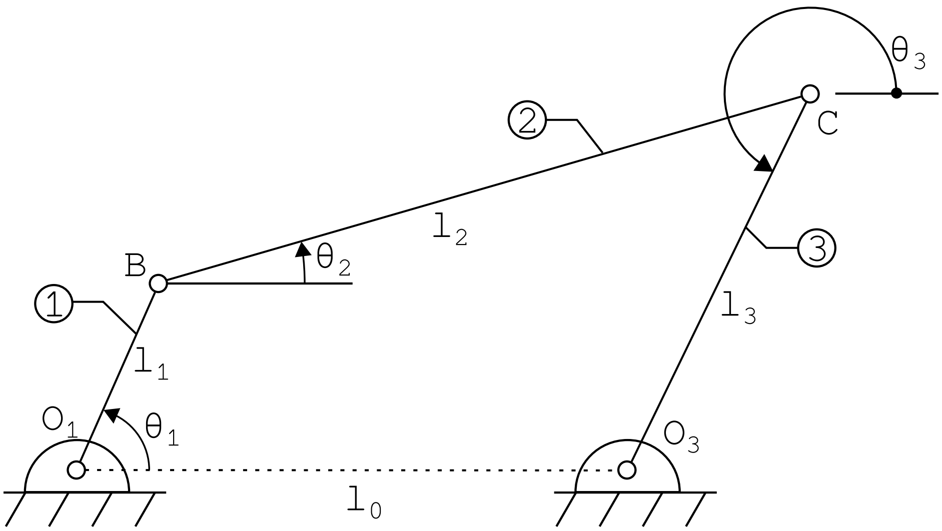


Figure 3: `Four-bar linkage`