ME 380
Kinematics and Machine Dynamics
Homework III Solutions

1. [15 points] Let r_1, r_2, and r_3 be vectors on a plane such that

$$r_1 + r_2 + r_3 = 0$$

If we additionally know that

$$\frac{r_{1y}}{r_{1x}} = 2, \quad \frac{r_{2y}}{r_{2x}} = \frac{1}{2}, \quad r_3 = \begin{bmatrix} 50 & 75 \end{bmatrix}^\top$$

Find r_1 and r_2 by using the vector cross product method.

Solution: By the cross product method, we have

$$l_1 = \frac{-r_3 \cdot \left(\hat{r}_2 \times \hat{k}\right)}{\hat{r}_1 \cdot \left(\hat{r}_2 \times \hat{k}\right)}, \qquad l_2 = \frac{-r_3 \cdot \left(\hat{r}_1 \times \hat{k}\right)}{\hat{r}_2 \cdot \left(\hat{r}_1 \times \hat{k}\right)} \tag{1}$$

where $\hat{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. We also know the directions of the vectors r_1 and r_2 as the ratio of their components are given. Thus,

$$\hat{r}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^\top, \qquad \hat{r}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^\top$$

Now we calculate

$$\begin{aligned} \hat{r}_1 \times \hat{k} &= (\hat{r}_1)_\times \hat{k} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ \hat{r}_2 \times \hat{k} &= (\hat{r}_2)_\times \hat{k} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \end{aligned}$$

Next, we have

$$\begin{aligned} -\hat{r}_1 \cdot \left(\hat{r}_2 \times \hat{k}\right) &= \hat{r}_2 \cdot \left(\hat{r}_1 \times \hat{k}\right) = \frac{3}{5} \\ -r_3 \cdot \left(\hat{r}_2 \times \hat{k}\right) &= 20\sqrt{5} \\ -r_3 \cdot \left(\hat{r}_1 \times \hat{k}\right) &= -5\sqrt{5} \end{aligned}$$

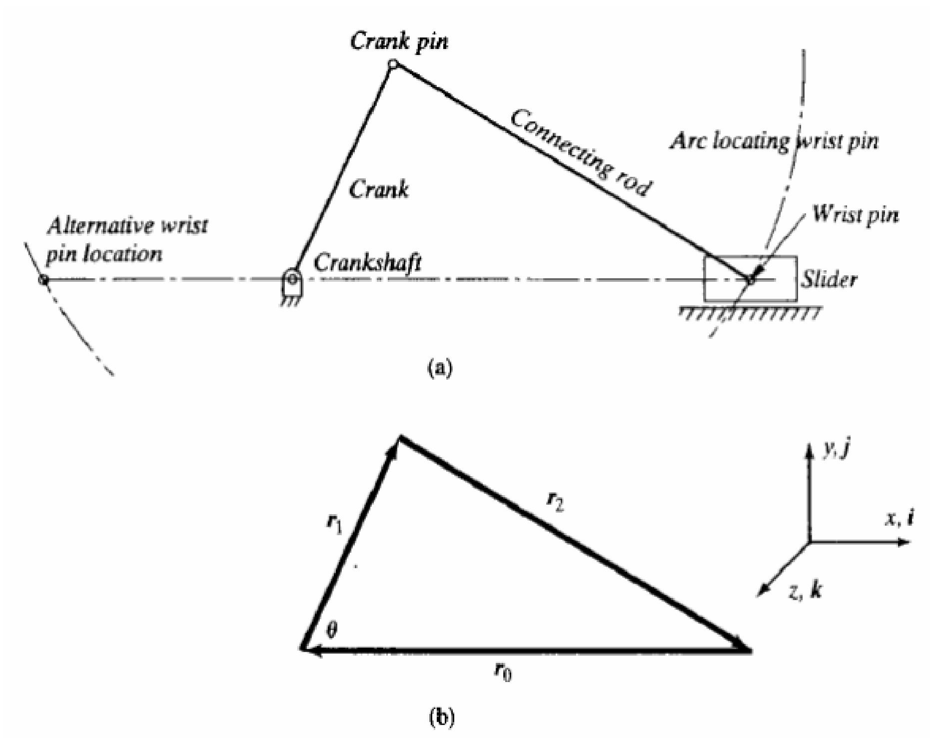
Plugging these back in (1) and calculating, we have

$$l_1 = \frac{-100\sqrt{5}}{3}, \qquad l_2 = \frac{-25\sqrt{5}}{3}$$

as a result

$$r_1 = l_1 \hat{r}_1 = -\frac{100}{3}\begin{bmatrix}1\\2\\0\end{bmatrix}, \qquad r_2 = l_2 \hat{r}_2 = -\frac{25}{3}\begin{bmatrix}2\\1\\0\end{bmatrix}.$$

2. [15 points] An inline slider-crank linkage has a crank length l_1 and a connecting rod length $l_2 = \frac{3}{2} l_1$. Find the connecting rod and the slider position (in terms of l_1) when the crank angle is $\theta = \frac{2\pi}{9}$ by using analytical vector methods.



(a) Slider-crank linkage. (b) Vector representation.

Solution: Let r_0 denote the vector from the wrist pin to the crankshaft, r_1 denote the vector from the crankshaft to the crank pin, and r_2 denote the vector from the crank pin to the wrist pin.

<u>Known</u>
r_1
$l_2 = 1.5 l_1$
$\hat{r}_0 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top$

<u>Unknown</u>
$\hat{r}_2$ (direction of r_2)
l_0 (magnitude of r_0)

This corresponds to case 4 in the cross product method with $a = r_0$, $b = r_2$, and $c = r_1$. We have

$$\begin{aligned} r_0 &= \left\{ -r_1 \cdot \hat{r}_0 \mp \sqrt{l_2^2 - \left[r_1 \cdot \left(\hat{r}_0 \times \hat{k} \right) \right]^2} \right\} \hat{r}_0 \\ r_2 &= - \left[r_1 \cdot \left(\hat{r}_0 \times \hat{k} \right) \right] \left(\hat{r}_0 \times \hat{k} \right) \pm \sqrt{l_2^2 - \left[r_1 \cdot \left(\hat{r}_0 \times \hat{k} \right) \right]^2} \hat{r}_0 \end{aligned}$$

Straightforward calculation (which the students are expected to get right) gives

$$\begin{aligned} r_0 &= l_1 \left(\cos\left(\frac{2\pi}{9} \right) \mp \sqrt{\frac{9}{4} - \sin\left(\frac{2\pi}{9} \right)^2} \right) \hat{r}_0 \\ r_2 &= -l_1 \left(\sin\left(\frac{2\pi}{9} \right) \hat{j} \pm \sqrt{\frac{9}{4} - \sin\left(\frac{2\pi}{9} \right)^2} \hat{r}_0 \right) \end{aligned}$$

3. [15 points] Plot the slider position (with Julia or Matlab) versus θ for an inline slider-crank linkage for which the ratio of the connecting rod to crank length is $\frac{3}{2}$. Let $\theta \in [0, 2\pi]$, discretized uniformly with step length $\delta\theta = \frac{\pi}{9}$.

Solution:

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using PyPlot

const l1 = 1

f(\theta) = 1*(cos(\theta) + sqrt((3/2)^2 - (sin(\theta))^2))*l1
\phi = 0:2\pi/9:2\pi

fig = plt[:figure]("Crank angle vs Slider Position")
plt[:clf]()
plt[:plot](\phi, f.(\phi), marker="d", markersize=5)
plt[:xlabel](L"$\theta$ [rad]", fontsize=15)
plt[:ylabel]("Slider position", fontsize=15)
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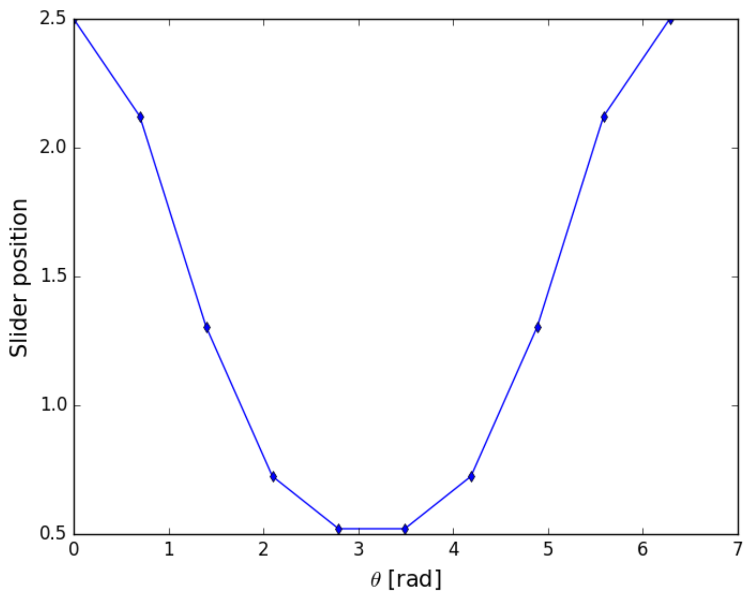

4. [15 points] The link lengths of a planar four-bar mechanism are $l_0 = 120$, $l_1 = 60$, $l_2 = 140$, $l_3 = 80$. Find the orientation of links 2 and 3 when the internal angle between the crank and the fixed link is $\frac{\pi}{6}$ and the linkage is in the open phase (i.e. the coupler does not cross the fixed link). Use the vector cross-product method.

Solution: We have the vector loop equation: $r_0 + r_1 + r_2 + r_3 = 0$, where $r_0 = -120\hat{i}$, $r_1 = 51.96\hat{i} + 30\hat{j}$. Thus, the diagonal vector and the unit vector along it are given by

$$\begin{aligned} r_d &= r_0 + r_1 = -68.04\hat{i} + 30\hat{j} \\ \hat{r}_d &= -0.915\hat{i} + 0.404\hat{j} \end{aligned}$$

Now, we can use the vector cross product method on the equation

$$r_2 + r_3 + r_d = 0$$

where the directions $\hat{r}_2$ and $\hat{r}_3$ are unknown. This corresponds to case 3 of the cross product method. After defining $\Delta = \frac{l_d^2 + l_3^2 - l2^2}{2l_d}$, we have

$$\begin{aligned} r_2 &= \mp\sqrt{l_3^2 - \Delta^2}\left(\hat{r}_d \times \hat{k}\right) + (\Delta - l_d)\,\hat{r}_d \\ r_3 &= \pm\sqrt{l_3^2 - \Delta^2}\left(\hat{r}_d \times \hat{k}\right) - \Delta\hat{r}_d \end{aligned}$$

Computing these quantities yield

$$
\begin{aligned}
r_2 &= (115.26 \mp 24.68)\hat{\imath} + (-50.89 \mp 55.91)\hat{\jmath} \\
r_3 &= (-47.26 \pm 24.68)\hat{\imath} + (20.86 \pm 55.91)\hat{\jmath}
\end{aligned}
$$

For open configuration assembly, the vector r_3 must have a negative $\hat{\jmath}$ measure number. Hence, we select the negative sign

$$r_3 = -71.94\hat{\imath} - 35.05\hat{\jmath}$$

Also for r_2, use consistent set of signs to get

$$r_2 = 139.94\hat{\imath} + 5.02\hat{\jmath}$$

5. [20 points (bonus)] *Properties of rotation matrices*
Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is, R satisfies $R = \exp(\omega_\times \theta) = I + (\sin\theta)\omega_\times + (1 - \cos\theta)\omega_\times^2$.
   (a) [5 points] Show that the eigenvalues of $\omega_\times$ are 0, i, and $-i$, where $i = \sqrt{-1}$. What is the eigenvector whose eigenvalue is 0?

Solution: Let's first establish that the eigenvalues of $\omega_\times$ are either 0 or purely imaginary.

$$\langle \omega_\times x, y\rangle = \langle x, \omega_\times^\top y\rangle = \langle x, -\omega_\times y\rangle = -\langle x, \omega_\times y\rangle$$

Suppose that $x \neq 0$ is an eigenvector with eigenvalue λ. Setting $y = x$, we have

$$\langle \omega_\times x, x\rangle = \langle \lambda x, x\rangle = \lambda \|x\|^2$$

On the other hand,

$$\langle \omega_\times x, x\rangle = -\langle x, \omega_\times x\rangle = -\langle x, \lambda x\rangle = -\bar{\lambda}\langle x, x\rangle = -\bar{\lambda}\|x\|^2$$

Therefore, we have that $\lambda + \bar{\lambda} = 0$. This implies that either $\lambda = 0$ or λ is purely imaginary. Now there exists an eigenvalue at 0 because $\omega_\times \omega = 0$ (i.e., ω is the eigenvector whose eigenvalue is 0). We know that the remaining eigenvalues are αi and $-\alpha i$ for some $\alpha \in \mathbb{R}$. It remains to show that $\alpha = 1$.
To that end, note that $\omega_\times v = \omega \times v$ and thus

$$\|\omega_\times v\| = \|\omega \times v\| = \|v\| \sin(\measuredangle(\omega, v))$$

where $\measuredangle(\omega, v)$ denotes the angle between ω and v. Notice that

$$\max_{v \in \mathbb{R}^3} \|v\| \sin\left(\measuredangle(\omega, v)\right) = \|v\|$$

As a result,

$$\sup_{v \in \mathbb{R}^3} \frac{\|\omega_\times v\|}{\|v\|} = 1$$

which implies that $|\lambda| \leq 1$. However, if $v \perp \omega$ this supremum is actually attained, so there exists an eigenvalue of $\omega_\times$ such that $|\lambda| = 1$. This cannot be the eigenvalue at 0, so it must be that $\alpha = 1$, which was to be demonstrated.

Note: A straightforward calculation would also produce the same result.

(b) [10 points] Show that the eigenvalues of R are 1, $e^{i\theta}$, and $e^{-i\theta}$. What is the eigenvector whose eigenvalue is 1?

Solution: Since $|R| = 1$, we have the following sequence of equalities

$$|I - R| = |R|\,|I - R| = \left|R^\top\right|\,|I - R| = \left|R^\top - I\right| = |R - I| = -\,|I - R|$$

where the last equality follows from the fact that 3 is odd. Thus, $|I - R| = 0$ and $\lambda_1 = 1$ is an eigenvalue of R.

Now, let $u_1 = \omega$ be the unit eigenvector of λ_1, so $Ru_1 = u_1$. We know that the matrix R is a rotation of an angle θ around this axis u_1. Let us form a new coordinate system using u_1, u_2, and $u_1 \times u_2$, where u_2 is a vector orthogonal to u_1, so the new system is right-handed. The transformation between the old coordinate system $\{e_1, e_2, e_3\}$ and the new system is given by a matrix B with column vectors u_1, u_2, and u_3. Thus we have that $Be_i = u_i$. Let us call Q, the matrix similar to R, through the transformation $Q = B^{-1}RB$. Then

$$Qe_1 = B^{-1}RBe_1 = B^{-1}Ru_1 = B^{-1}u_1 = e_1.$$

The fact that $Qe_1 = e_1$ has two meanings

- The first column of Q is $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$
- The first row of Q is $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$.

Then Q is a matrix of the type

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$$

Since R is orthogonal, so is Q. Thus the vectors $\begin{bmatrix} a & c \end{bmatrix}^\top$ and $\begin{bmatrix} b & d \end{bmatrix}^\top$ are orthogonal. Moreover, since

$$1 = |R| = |Q| = ad - bc$$

we have that the minor matrix with entries a, b, c, d is a rotation (orthogonal with determinant 1). Rotations in 2D are of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then,

$$Q = B^{-1}RB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

But we know that the eigenvalues of $B^{-1}RB$ are the same as those of R. We find the eigenvalues of the former matrix through its characteristic equation

$$|\lambda I - Q| = 0 \implies (\lambda - 1)\left((\lambda - \cos(\theta))^2 + \sin(\theta)^2\right) = 0$$

which implies that either $\lambda = 1$, or

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = \cos(\theta) \pm i\,\sin(\theta) = e^{\pm i\theta}$$

Note: A straighforward calculation would also produce the same result, but would be much more tedious.

(c) [5 points] Let $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$ be a rotation matrix. We know that $|R| = r_1 \cdot (r_2 \times r_3)$. Show that $r_1 \cdot (r_2 \times r_3) = r_2 \cdot (r_3 \times r_1) = r_3 \cdot (r_1 \times r_2)$.

Solution: On one hand we have,

$$r_1 \cdot (r_2 \times r_3) = r_1^\top (r_2)_\times r_3 = -r_3^\top (r_2)_\times r_1 = -r_3 \cdot (r_2 \times r_1) = r_3 \cdot (r_1 \times r_2)$$

where the second equality follows from transposing the whole expression (this is a scalar so transposing does not affect the result) and noting that the transpose of a skew-symmetric matrix is equal to its negation. On the other hand,

$$r_1 \cdot (r_2 \times r_3) = r_1^\top (r_2)_\times r_3 = -r_1^\top (r_3)_\times r_2 = r_2^\top (r_3)_\times r_1 = r_2 \cdot (r_3 \times r_1)$$