ME 380
Kinematics and Machine Dynamics
Homework II Solutions

1. [15 points] For each of the following set of link lengths

   (i) $l_0 = 1$, $l_1 = 3.25$, $l_2 = 1.5$, $l_3 = 3.5$

   (ii) $l_0 = 2.5$, $l_1 = 1$, $l_2 = 2.5$, $l_3 = 2$

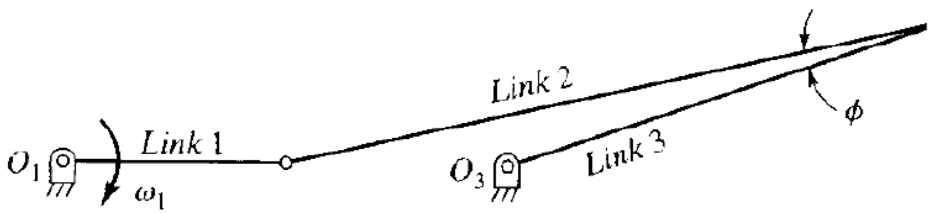


Figure 1 – Four–bar Mechanism

   (a) [5 points] Determine whether the links can actually form a mechanism with the dimensions given.

   Solution: $l_{max} > l_{min} + l_a + l_b$: this may **not** happen.

   (i) $3.5 < 3.25 + 1.5 + 1$, so it is a mechanism.

   (ii) $2.5 < 2.5 + 1 + 2$, so it is a mechanism.

   (b) [5 points] If a mechanism exists, identify the motion relationships (crank-rocker, drag-link, double-rocker, triple-rocker or change point)

   Solution:

   (i) $l_{max} + l_{min} < l_a + l_b$ and since $l_0 = l_{min}$, it is a drag-link mechanism.

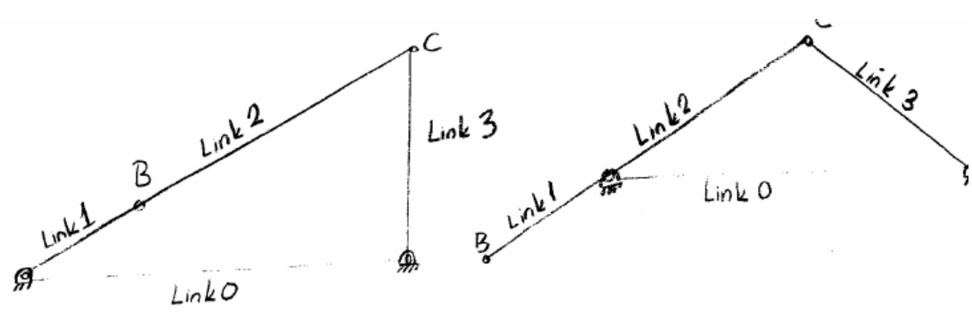
   (ii) $l_{max} + l_{min} < l_a + l_b$ and since $l_1 = l_{min}$, it is a crank-rocker mechanism.

   (c) [5 points] Show the limiting positiins if the mechanism is a crank-rocker, double-rocker, or triple-rocker mechanism.

   Solution:

   (i) No limiting position.

(ii)



2. [18 points] Find the range of values of the unknown link length, l_3, for $l_0 = 120$, $l_1 = 220$, $l_2 = 80$, if the linkage will theoretically act as

   (a) [3 points] a crank-rocker mechanism (where link 1 rotates through 2π)

   Solution: This scheme cannot be achieved because l_1 cannot be the shortest link.

   (b) [3 points] a drag-link mechanism

   Solution: This scheme cannot be achieved because l_0 cannot be the shortest link.

   (c) [3 points] a double-rocker mechanism

   Solution:
   $$l_{max} + l_{min} < l_a + l_b, \qquad l_2 = l_{min}$$

   If $l_3 = l_{max}$ then $l_3 > 220$, and $l_{max} + 80 < 120 + 220$. Thus $220 < l_3 < 260$

   If $l_3 \neq l_{max}$ then $l_3 < 220$ and $220 + 80 < l_3 + 120$. Thus $180 < l_3 < 220$

   Putting these together: $180 < l_3 < 260$.

   (d) [3 points] a change-point mechanism

   Solution:
   $$l_{max} + l_{min} = l_a + l_b$$

   If $l_3 = l_{max}$ then $l_{max} + 80 = 120 + 220$. Thus $l_3 = 260$

   If $l_3 \neq l_{max}$ then $220 + 80 = l_3 + 120$. Thus $l_3 = 180$

(e) [3 points] a Grashof mechanism

Solution:

$$l_{max} + l_{min} \leq l_a + l_b$$

Double-rocker	$180 < l_3 < 260$
Change-Point	$l_3 = 180$ or $l_3 = 260$

(f) [3 points] a triple-rocker mechanism

Solution:

$$l_{max} + l_{min} > l_a + l_b$$

If $l_3 = l_{max}$ then
$l_3 < 120 + 220 + 80 \;\Rightarrow\; l_3 < 420$ and
$l_3 + 80 < 120 + 220 \;\Rightarrow\; l_3 > 260$, thus
$260 < l_3 < 420$.

If $l_3 \neq l_{max}$ then
$220 + 80 > l_3 + 120 \;\Rightarrow\; l_3 < 180$, thus
$0 < l_3 < 180$.

But to ensure a mechanism, we need to satisfy $\triangle$-inequality:
$220 - 120 - 80 < l_3 \implies l_3 > 20$. Thus $20 < l_3 < 180$.

3. [15 points] Let $l_0 = 150$mm, $l_1 = 110$mm, $l_2 = 150$mm, find the length of the link l_3 so that the linkages form a crank-rocker mechanism and the transmission angle falls between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Solution: Grashof mechanisms satisfy: $l_{max} + l_{min} \leq l_a + l_b$.

If l_3 is not maximum

$$\begin{aligned} l_0 + l_1 &< l_2 + l_3 \\ 150 + 110 - 150 &< l_3 \\ l_3 &> 110\text{mm}. \end{aligned}$$

If l_3 is maximum

$$\begin{aligned} l_3 + l_1 &< l_2 + l_0 \\ l_3 &< 150 + 150 - 110 \\ l_3 &< 190\text{mm}. \end{aligned}$$

Thus we have $110\text{mm} < l_3 < 190\text{mm}$.

$$\begin{aligned} \cos(\phi_{min}) &= \frac{l_2^2 + l_3^2 - (l_0 - l_1)^2}{2 l_2 l_3} \\ \cos\left(\frac{\pi}{4}\right) &= \frac{150^2 + l_3^2 - (150 - 110)^2}{2 \cdot 150 \cdot l_3} \\ l_3^2 - 212.13 l_3 + 20900 &= 0 \\ l_3 &= 106.065 \pm j98.235 \end{aligned}$$

$$\begin{aligned} \cos(\phi_{max}) &= \frac{l_2^2 + l_3^2 - (l_0 + l_1)^2}{2 l_2 l_3} \\ \cos\left(\frac{3\pi}{4}\right) &= \frac{150^2 + l_3^2 - (110 + 150)^2}{2 \cdot 150 \cdot l_3} \\ l_3^2 + 212.13 l_3 - 45100 &= 0 \\ l_3 &\in \{-343.4461, 131.316\} \end{aligned}$$

For $\phi_{min} = \frac{\pi}{4}$ we cannot find a logical link length so there is no answer.
For $\phi_{max} = \frac{3\pi}{4}$, the link length should be $l_3 = 131.316$mm.

4. [15 points] This problem refers to an offset slider-crank mechanism depicted on the next page.

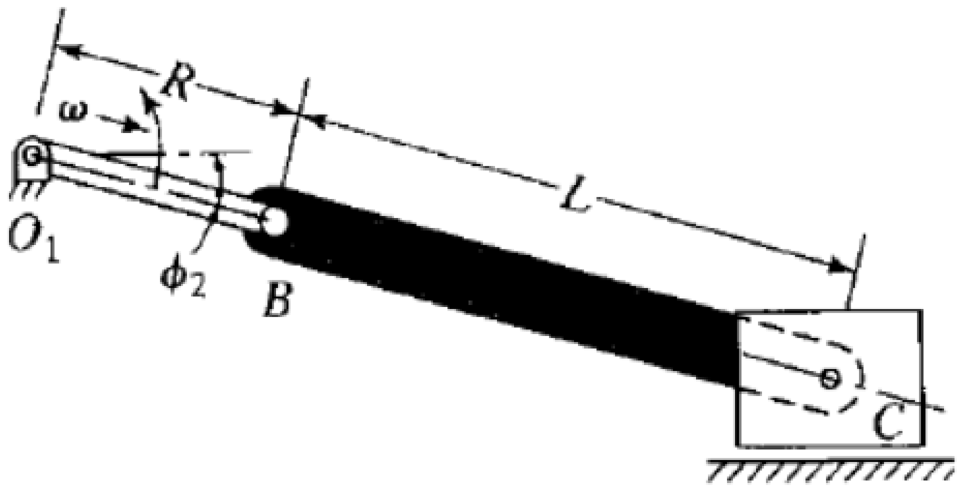


Figure 2 – Offset Slider–Crank Mechanism

The transmission angle is limited to the range between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ (i.e., the angle between the connection rod and the slider path must fall between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$). The crank length is 400mm, and the offset is 200mm; find the minimum connecting rod length.

Solution: Calculus approach
Looking at a general offset slider-crank mechanism in Figure 1, it needs to satisfy the following two kinematic equations – the components of the loop equation in the world x and y directions

$$\begin{aligned} R\cos(\theta) + L\sin\varphi - s &= 0, \\ R\sin(\theta) - L\cos\varphi + E &= 0. \end{aligned}$$

The latter equation can be solved for L to give

$$L(\theta, \varphi) = \frac{R\sin\theta + E}{\cos\varphi}. \tag{1}$$

We want to maximize over θ the minimum over φ of L. Any smaller value of L will yield a transmission angle strictly less than $\frac{\pi}{4}$ at an appropriate value of θ. The maximizer θ^* needs to satisfy the first-order necessary condition

$$\frac{\partial L}{\partial \theta}(\theta^*, \varphi) \;= \frac{R\cos\theta^*}{\cos\varphi} = 0,$$

the solution of which is $\theta^* = \frac{\pi}{2}$ or $\theta^* = \frac{3\pi}{2}$. Plugging $\theta^* = \frac{3\pi}{2}$ into (1) gives a negative value for L, and so is infeasible. As a result, $\theta^* = \frac{\pi}{2}$, so we have

$$L(\theta^*, \varphi) \;= L\left(\frac{\pi}{2}, \varphi\right) = \frac{R+E}{\cos\varphi}$$

Within the allowed range for φ, (i.e., $\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$), this function gets minimized when $\varphi = \varphi^* = \frac{\pi}{4}$, yielding

$$L^* \;= L(\theta^*, \varphi^*) = \frac{(R+E)}{\cos\left(\frac{\pi}{4}\right)} = (R+E)\sqrt{2} = 600\sqrt{2}.$$

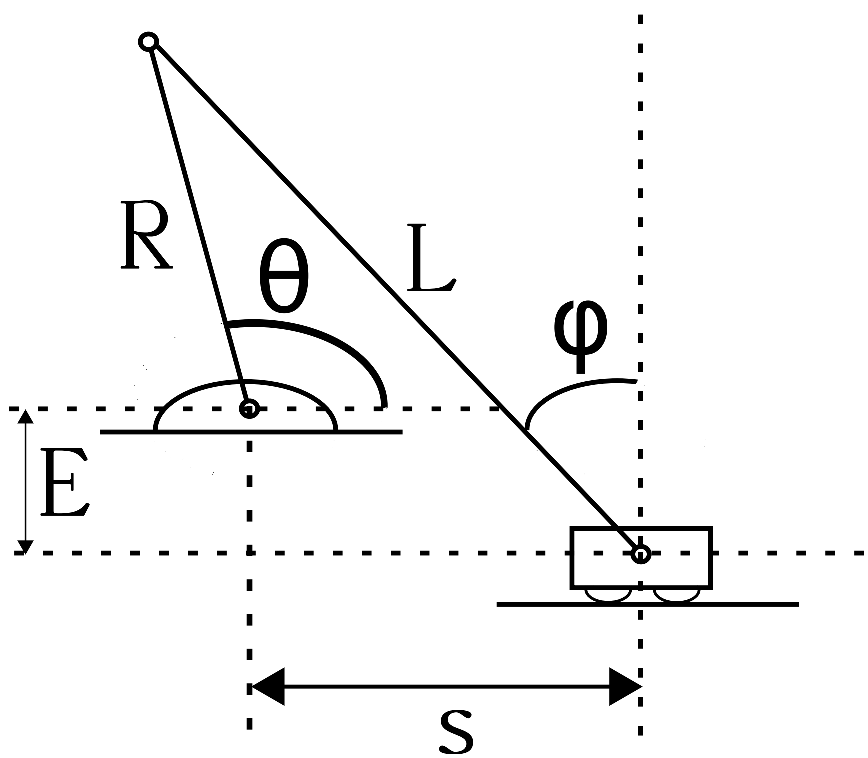
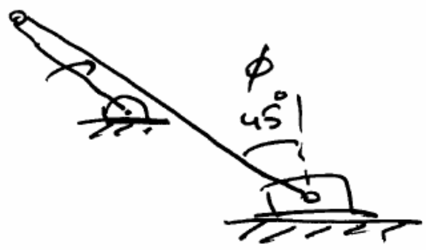


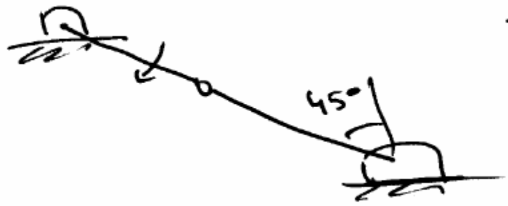
Figure 1: `Offset Slider-Crank Mechanism`

Solution: Geometric approach

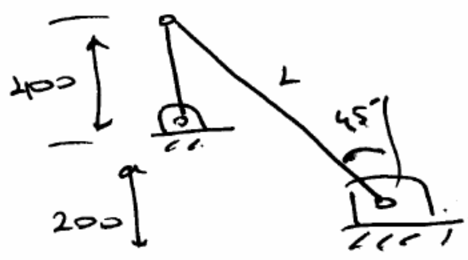
Study the slider crank linkage at different configurations:



→ in this case whe when driver crank starts rotating CW, ϕ will decrease & become less than 45°.



→ in this case when driver crank rotates slider will lose contact.



→ this case will ensure the right solution

$$L = \frac{(400+200)}{\cos 45}$$

$$L = 600\sqrt{2} \text{ mm}$$

5. [20 points] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 - \alpha^2 \\ x_1 x_2 - \beta \end{bmatrix} \quad (2)$$

parameterized by $\alpha > 0$ and $\beta > 0$. Define $x_0 := \begin{bmatrix} \frac{2}{8} & \frac{3}{8} \end{bmatrix}^\top$.

Solution:

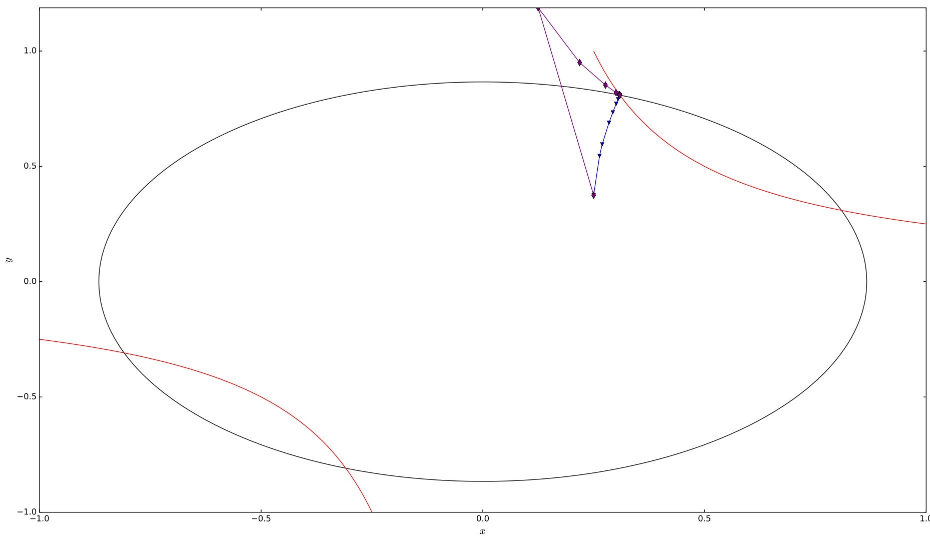


Figure 2: `Convergence of Newton's Method:` The black curve is the set $f_1(x) = x_1^2 + x_2^2 - \alpha^2 = 0$ and the red curve is the set $f_2(x) = x_1 x_2 - \beta = 0$. The intersections denote the solution where both $f_1(x) = f_2(x) = 0$. For the starting position given in the question, Newton's algorithm converges to the solution shown shown in blue and purple iterates.

(a) [10 points] Suppose $(\alpha, \beta) = (\frac{\sqrt{3}}{2}, \frac{1}{4})$.
Use either Matlab or Julia to code the Newton-Raphson algorithm. Use this algorithm to find an x^* such that $f(x^*) \approx 0$ starting from the initial guess x_0. You can assume the algorithm has "converged" whenever $\|f(x^*)\| \leq \epsilon = 10^{-6}$.

Solution:

```
const \alpha = sqrt(3)/2
const \beta = 1/4
const \zeta = 0.8 # damping factor

const absTol = 1e-6
const relTol = 1e-6

x0 = [2; 3]./8 # Initial guess

fvec = zeros(2,1)
fjac = zeros(2,2)
```

```
function f!(x, fvec)
    fvec[1] = sum(x.^2)- \alpha^2
    fvec[2] = prod(x) - \beta
end

function df!(x, fjac)
    fjac[1,1] = 2x[1]
    fjac[1,2] = 2x[2]
    fjac[2,1] = x[2]
    fjac[2,2] = x[1]
end

f!(x0, fvec)
df!(x0, fjac)
\deltax = -fjac\fvec
xstar = x0
xhistory = x0'

while norm(fvec) > absTol || norm(\deltax) > relTol
    xstar += \deltax
    xhistory = cat(xhistory, xstar'; dims=1)
    f!(xstar, fvec)
    df!(xstar, fjac)
    \deltax = -\zeta*(fjac\fvec)
end
```

(b) [10 points] Find the analytic solutions of the equation $f(x) = 0$ using algebra. Compare your solutions with the solution you obtained in part (a).

Solution: By combining the functions in (2), a solution to $f(x) = 0$ will satisfy

$$(x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 = \alpha^2 + 2\beta \implies x_1 + x_2 = \pm\sqrt{\alpha^2 + 2\beta}$$

Define $\eta = \sqrt{\alpha^2 + 2\beta}$. We have either $x_2 = -x_1 + \eta$ or $x_2 = -(x_1 + \eta)$. We successively input these conditions to the equation $f_2(x) = 0$ to get

$\underline{x_2 = -x_1 + \eta}$

$$\begin{aligned} x_1x_2 &= x_1(-x_1 + \eta) = \beta \\ \implies & x_1^2 - \eta x_1 + \beta = 0 \\ \implies & x_1 = \frac{1}{2}\left(\eta \pm \sqrt{\eta^2 - 4\beta}\right) \end{aligned}$$

$\underline{x_2 = -(x_1 + \eta)}$

$$\begin{aligned} x_1x_2 &= -x_1(x_1 + \eta) = \beta \\ \implies & x_1^2 + \eta x_1 + \beta = 0 \\ \implies & x_1 = \frac{1}{2}\left(-\eta \pm \sqrt{\eta^2 - 4\beta}\right) \end{aligned}$$

Substituting $(\alpha, \beta) = (\frac{\sqrt{3}}{2}, \frac{1}{4})$, we get all four solutions to $f(x) = 0$:

x_1	0.809017	0.309017	−0.309017	−0.809017
x_2	0.309017	0.809017	−0.809017	−0.309017

Table 1: Analytic solution of the equations

(c) [5 points (bonus)] Use Matlab's `fsolve` or Julia's `nlsolve` command (contained in the package `NLsolve`) to find an x^* such that $f(x^*) \approx 0$ starting from the initial guess x_0. Compare the result with what you found in part (a).

Solution:

```
using NLsolve

const \alpha = sqrt(3)/2
const \beta = 1/4

x0 = [2; 3]./8 # Initial guess

fvec = zeros(2,1)
fjac = zeros(2,2)

function f!(x, fvec)
    fvec[1] = sum(x.^2)- \alpha^2
    fvec[2] = prod(x) - \beta
end

function df!(x, fjac)
    fjac[1,1] = 2x[1]
    fjac[1,2] = 2x[2]
    fjac[2,1] = x[2]
    fjac[2,2] = x[1]
end

solNL = nlsolve(f!, df!, x0)
xNLs = solNL.zero
```