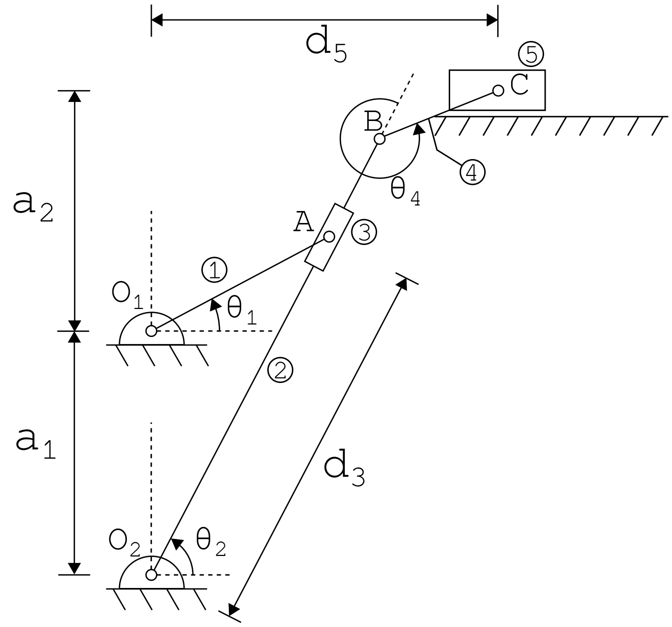
ME 380: Kinematics and Machine Dynamics
Final Examination Solutions

Question 1 .... *0 (main) + 10 (bonus) points*

Consider the Whitworth quick-return mechanism shown the figure below. The lengths of links 1, 2, and 4 are given by l_1, l_2, and l_4. The position of link 3 along link 2 is given by d_3 and the horizontal position of link 5 is given by d_5.

Quantity	Length [m]
a_1	1.0
a_2	1.0
l_1	$\frac{3}{4}$
l_2	$\frac{7}{4}$
l_4	$\frac{2}{3}$



(a) Given θ_1, write down the loop equations used to solve for θ_2, d_3, θ_4, and d_5. Express the vector form of the equations in the world ($\Sigma_0 = \{x_0, y_0, z_0\}$) frame.

Solution: We need two vector loop equations in order to solve for 4 scalar variables. These loop equations are

$$
\begin{aligned}
p^{O_1A_1} + r - p^{O_2A_2} + a_1 y_0 &= 0, \\
p^{O_2B} + p^{BC} - d_5 x_0 - (a_1 + a_2) y_0 &= 0.
\end{aligned} \tag{1}
$$

where $r = \overrightarrow{A_1A_2}$. Here A_1 and A_2 are points coincident with point A at the instant in consideration with point A_1 fixed on link 1 and A_2 fixed on link 2. Writing out these equations in the world frame Σ_0 gives

$$
\begin{aligned}
l_1 \cos(\theta_1) - d_3 \cos(\theta_2) &= 0, \\
l_1 \sin(\theta_1) - d_3 \sin(\theta_2) + a_1 &= 0, \\
l_2 \cos(\theta_2) + l_4 \cos(\theta_2 + \theta_4) - d_5 &= 0, \\
l_2 \sin(\theta_2) + l_4 \sin(\theta_2 + \theta_4) - (a_1 + a_2) &= 0.
\end{aligned}
$$

(b) Given ${}^0\omega^1 = \dot{\theta}_1 z_0$, write down the equations to find ${}^0\omega^2 = \dot{\theta}_2 z_0$, $\dot{d}_3$, ${}^0\omega^4 = \dot{\theta}_4 z_0$, and $\dot{d}_5$ in matrix form.

Solution: Differentiating equation (1) in the 0^{th} frame gives

$$\begin{aligned} {}^0\omega^1 \times p^{O_1A_1} - {}^2v^{A_1} - {}^0\omega^2 \times r - {}^0\omega^2 \times p^{O_2A_2} &= 0, \\ {}^0\omega^2 \times p^{O_2B} + {}^0\omega^4 \times p^{BC} - \dot{d}_5 x_0 &= 0. \end{aligned} \tag{2}$$

Writing out these equations in the world frame Σ_0 gives $J\dot{q} = 0$, where $q = (\theta_1, \theta_2, d_3, \theta_4, d_5)$. Notice the block diagonal structure of the matrix J.

$$\begin{bmatrix} -l_1 \sin\theta_1 & d_3 \sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ l_1 \cos\theta_1 & -d_3 \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & -l_2 \sin\theta_2 - l_4 \sin(\theta_2 + \theta_4) & 0 & -l_4 \sin(\theta_2 + \theta_4) & -1 \\ 0 & l_2 \cos\theta_2 + l_4 \cos(\theta_2 + \theta_4) & 0 & l_4 \cos(\theta_2 + \theta_4) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{d}_5 \end{bmatrix} = 0. \tag{3}$$

(c) Given ${}^0\alpha^1 = \ddot{\theta}_1 z_0$, write down the equations to find ${}^0\alpha^2 = \ddot{\theta}_2 z_0$, $\ddot{d}_3$, ${}^0\alpha^4 = \ddot{\theta}_4 z_0$, and $\ddot{d}_5$ (You can leave the equations in vector form, i.e., no need to write out the matrix form of the equations).

Solution: Differentiating equation (2) in the world frame Σ_0 gives

$$\begin{aligned} &{}^0\alpha^1 \times p^{O_1A_1} + {}^0\omega^1 \times \left({}^0\omega^1 \times p^{O_1A_1}\right) - {}^2a^{A_1} - {}^0\alpha^2 \times r - {}^0\omega^2 \\ &\quad \times \left({}^0\omega^2 \times r\right) - 2\,{}^0\omega^2 \times {}^2v^{A_1} - {}^0\alpha^2 \times p^{O_2A_2} - {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2A_2}\right) = 0, \\ &{}^0\alpha^2 \times p^{O_2B} + {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2B}\right) + {}^0\alpha^4 \times p^{BC} + {}^0\omega^4 \times \left({}^0\omega^4 \times p^{BC}\right) - \ddot{d}_5 x_0 = 0. \end{aligned} \tag{4}$$

(d) [10 points (bonus)] If, at this instant, the position-level kinematic analysis is performed with the solution $(\theta_1, \theta_2, d_3, \theta_4, d_5) = (30°, 64.715°, 1.521, -25.923°, 1.267)$, and $\dot{\theta}_1 = -\frac{1}{2}\,\frac{\text{rad}}{\text{s}}$, find $\dot{d}_5$.

Solution: Plugging given values in equation (3) and solving the linear equations, we get the solution

$$\begin{bmatrix} \dot{\theta}_2 & \dot{d}_3 & \dot{\theta}_4 & \dot{d}_5 \end{bmatrix} = \begin{bmatrix} -0.2027 & -0.2136 & 0.4943 & 0.1990 \end{bmatrix},$$

where the units of the angular and linear velocities are radians per second and meters per second, respectively.

Question 2 . . . . *0 (main) + 0 (bonus) points*

Consider the toggle linkage shown in Figure 1. The lengths of the links 1 through 4 are given by $l_1, \ldots, l_4$. The position vector from O_1 to O_2 is given by $p^{O_1O_2} = \begin{bmatrix} a_{0x} & -a_{0y} & 0 \end{bmatrix}^\top$, expressed in Σ_0. The masses of all of the links are uniformly distributed. Gravity acts in the downward direction within the plane of operation. Assume that the center of mass of link 5 is at the point B. In your solutions, make sure to use the numbering and naming convention given in Figure 1.

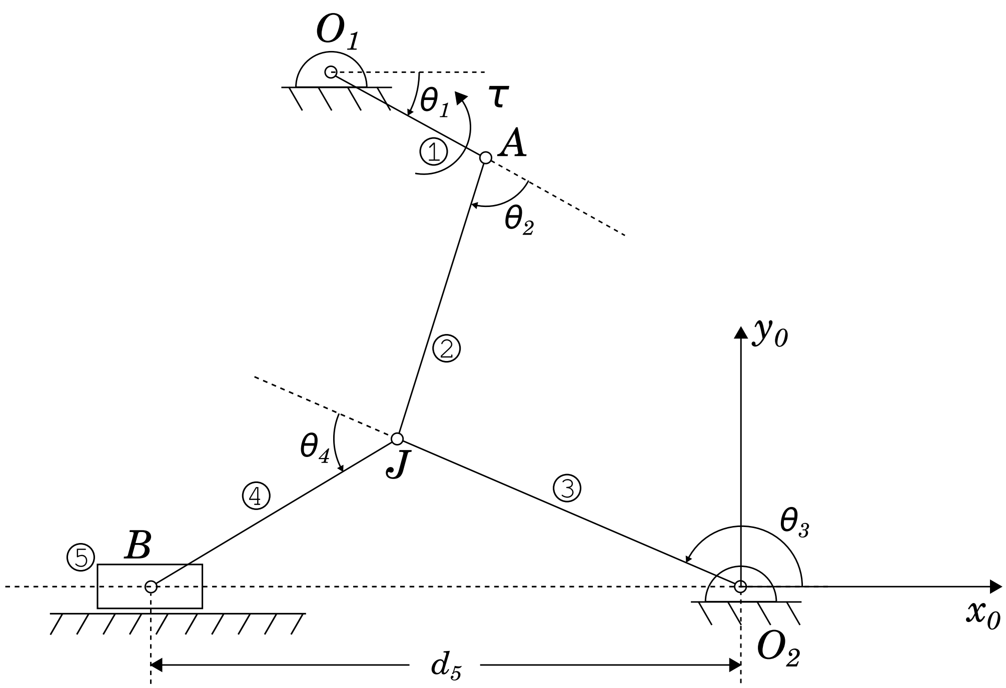


Figure 1: `Toggle Linkage`

(a) Given θ_1, write down the loop equations, used to solve for θ_2, θ_3, θ_4, and d_5. Express the vector form of the equations in the world frame (Σ_0).

Solution: Let $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AJ}$, $p_3 = \overrightarrow{JO_2}$, $p_4 = \overrightarrow{JB}$, $p_5 = \overrightarrow{O_2B}$.
The loop equations are given in vector form as

$$\begin{aligned} \sum_{i=1}^{3} p_i - p^{O_1O_2} &= 0, \\ -p_3 + p_4 - p_5 &= 0. \end{aligned} \tag{5}$$

Expressing these in Σ_0 gives

$$\begin{aligned} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) - l_3 \cos\theta_3 - a_{0x} &= 0, \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) - l_3 \sin\theta_3 + a_{0y} &= 0, \\ l_3 \cos\theta_3 + l_4 \cos(\theta_3 + \theta_4) - d_5 &= 0, \\ l_3 \sin\theta_3 + l_4 \sin(\theta_3 + \theta_4) &= 0. \end{aligned}$$

(b) Given ${}^0\omega^1 = \dot\theta_1 z_0$, write down the equations to find $\dot\theta_2$, $\dot\theta_3$, $\dot\theta_4$, and $\dot d_5$ in matrix form.

Solution: Differentiating equations (5) w.r.t. time yields

$$\begin{aligned} \sum_{i=1}^{3} {}^0\omega^i \times p_i &= 0, \\ -{}^0\omega^3 \times p_3 + {}^0\omega^4 \times p_4 - {}^0v^B &= 0. \end{aligned} \tag{6}$$

Expressing this in Σ_0 gives $J\dot q = 0$, i.e.,

$$\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & l_3 s_3 & 0 & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & -l_3 c_3 & 0 & 0 \\ 0 & 0 & -l_3 s_3 - l_4 s_{34} & -l_4 s_{34} & -1 \\ 0 & 0 & l_3 c_3 + l_4 c_{34} & l_4 c_{34} & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \\ \dot\theta_4 \\ \dot d_5 \end{bmatrix} = 0.$$

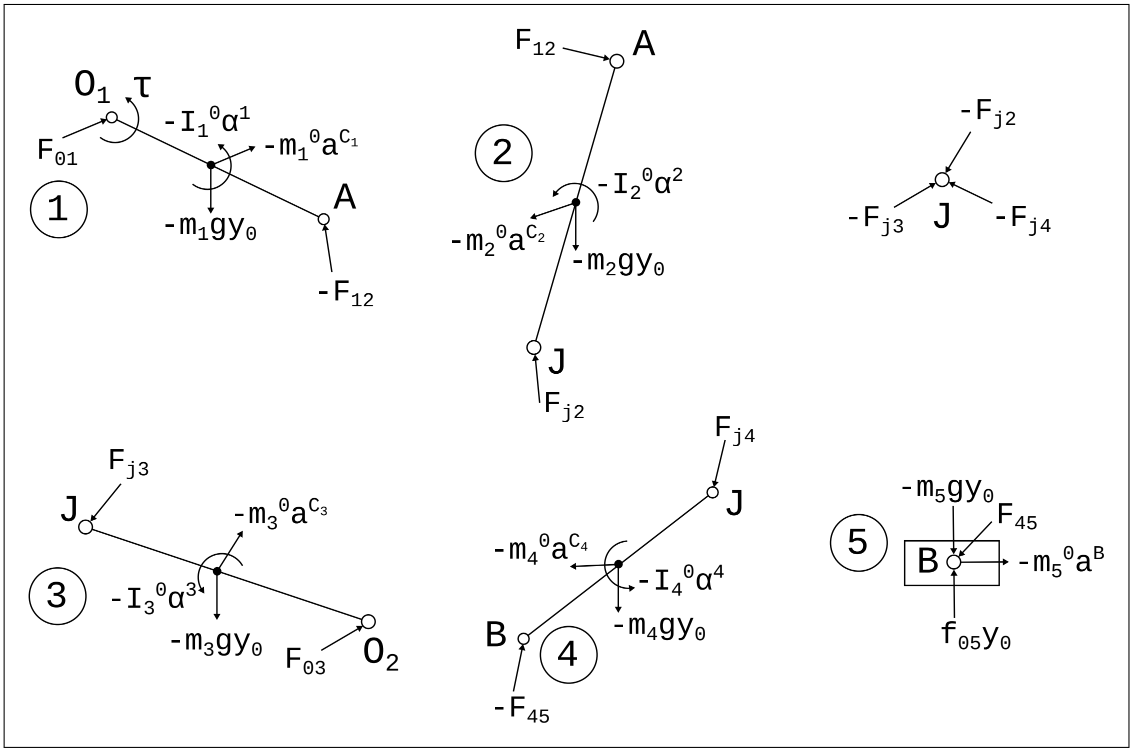
(c) Write down the equations that relate ${}^0\alpha^1$, ${}^0\alpha^2$, ${}^0\alpha^3$, ${}^0\alpha^4$, and ${}^0a^B$. (You can leave the equations in vector form, i.e., no need to write out the matrix form of the equations).

Solution: Differentiating equations (6) w.r.t. time gives

Acc.level kinematics	$\sum_{k=1}^{3} {}^0\alpha^k \times p_k + {}^0\omega^k \times \left({}^0\omega^k \times p_k\right) = 0,$ $-{}^0\alpha^3 \times p_3 - {}^0\omega^3 \times \left({}^0\omega^3 \times p_3\right) + {}^0\alpha^4 \times p_4 - {}^0\omega^4 \times \left({}^0\omega^4 \times p_4\right) - {}^0a^B = 0.$
	4 equations

(d) Draw the free body diagram of each rigid body, marking all the external forces and moments as well as the inertial forces and moments. (Hint: To simplify the free-body diagrams at the multiple link junction, consider the point J as an auxiliary rigid body with no mass.)

Solution:



(e) *Forward dynamics* is the problem of finding the link accelerations ${}^0\alpha^k,\ k = 1, \ldots, 4$ and ${}^0a^B$, given the input torque, τ. Derive the Newton-Euler equations for forward dynamics in vector form. There is **no need** to write out the matrix form of the equations. Clearly state what the unknowns are and how many of them there are. Count the number of equations you have derived to solve for these unknowns.

Solution: Newton-Euler forward dynamics

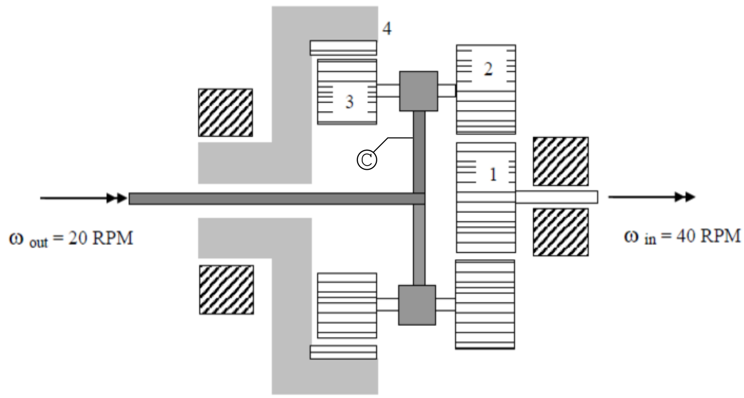
Dynamic force balance	Dynamic moment balance
$F_{01} - F_{12} - m_1 g y_0 - m_1 {}^0a^{C_1} = 0,$	$p^{C_1O_1} \times F_{01} - p^{C_1A} \times F_{12} + \tau z_0 - I_1 {}^0\alpha^1 = 0,$
$F_{12} - F_{j2} - m_2 g y_0 - m_2 {}^0a^{C_2} = 0,$	$p^{C_2A} \times F_{12} + p^{C_1J} \times F_{j2} - I_2 {}^0\alpha^2 = 0,$
$F_{j3} + F_{03} - m_3 g y_0 - m_3 {}^0a^{C_3} = 0,$	$p^{C_3J} \times F_{j3} + p^{C_3O_2} \times F_{03} - I_3 {}^0\alpha^3 = 0,$
$F_{j4} - F_{45} - m_4 g y_0 - m_4 {}^0a^{C_4} = 0,$	$p^{C_4J} \times F_{j4} - p^{C_4B} \times F_{45} - I_4 {}^0\alpha^4 = 0,$
$F_{45} - f_{05} y_0 - m_5 g y_0 - m_5 {}^0a^B = 0,$	$--$
$F_{j2} + F_{j3} + F_{j4} = 0.$	$--$
12 equations	4 equations

The acceleration-level kinematics from part (c) provides 4 more equations to work with.

Unknowns:	$F_{01}, F_{12}, F_{j2}, F_{j3}, F_{03}, F_{j4}, F_{45}, f_{05}, {}^0\alpha^1, {}^0\alpha^2, {}^0\alpha^3, {}^0\alpha^4, {}^0a^B$
	$2+2+2+2+2+2+2+1+1+1+1+1+1 = 20.$

Question 3 . . . . *0 (main) + 0 (bonus) points*

The ring gear in the epicyclic geartrain shown below is *not* fixed to the ground. The tooth numbers are $N_1 = 27$, $N_2 = 36$, $N_3 = 24$, $N_4 = 87$. Find the angular velocity of the ring gear. (Hint: Gears 2 and 3 are parts of the same rigid body.)



Solution: Let ${}^0\omega^C = \omega_C z_0$. We are given that $\omega_C = 20\,\text{rpm}$ and $\omega_1 = 40\,\text{rpm}$. Let P_1 and P_2 be the pitch point(s) between gears 1 and 2 with P_1 fixed on gear 1 and P_2 fixed on gear 2. We have

$$ {}^0v^{P_1} = \omega_1 r_1 y_0 = (\omega_C(r_1 + r_2) - \omega_2 r_2)\, y_0 = {}^0v^{P_2}. \tag{7}$$

Let Q_3 and Q_4 be the pitch point(s) between gears 3 and 4 with Q_3 fixed on gear 3 and Q_4 fixed on gear 4. We have

$$ {}^0v^{Q_3} = (\omega_C(r_1 + r_2) + \omega_3 r_3)\, y_0 = \omega_4 r_4 y_0 = {}^0v^{Q_4}. \tag{8}$$

Additionally, we know that $\omega_2 = \omega_3$ since these gears are physically mounted on the same rigid body. Solving equation (7) for $\omega_2 r_3$, we find

$$\omega_2 r_3 = \omega_C \frac{(r_1 + r_2) r_3}{r_2} - \omega_1 \frac{r_1}{r_2},$$

Plugging this back in equation (8) as well as $\omega_3 = \omega_2$, we have

$$\omega_4 r_4 = \omega_C \left(r_1 + r_2 + r_3 + \frac{r_1 r_3}{r_2} \right) - \omega_1 \frac{r_1 r_3}{r_2},$$

Now, notice that $r_4 = r_1 + r_2 + r_3$, so the above expression simplifies to

$$\omega_4 = \omega_C \left(1 + \frac{r_1}{r_2} \frac{r_3}{r_4} \right) - \omega_1 \frac{r_1}{r_2} \frac{r_3}{r_4} = \omega_C \left(1 + \frac{N_1}{N_2} \frac{N_3}{N_4} \right) - \omega_1 \frac{N_1}{N_2} \frac{N_3}{N_4}.$$

Plugging in the numerical values for N_1, N_2, N_3, N_4, ω_C and ω_1 yields $\boxed{\omega_4 = 15.862\,\text{rpm}}$.

Question 4 .... *0 (main) + 0 (bonus) points*

A CEP (critical extreme position) cam is being designed to provide the following motion to a translating follower

Follower Displacement		Cam Rotation [rad]
Rise	0 cm to 20 cm	0 rad to $\frac{2\pi}{3}$ rad
Fall	20 cm to 0 cm	$\frac{2\pi}{3}$ rad to $\frac{4\pi}{3}$ rad
Dwell	at 0 cm	$\frac{4\pi}{3}$ rad to 2π rad

Find a matrix representations for the coefficients of a **single** polynomial displacement function for the rise and fall portion of the follower. Clearly state your boundary conditions at the beginning of your solution.

Solution: The fundamental law of cam design states that the cam displacement function must at least be twice differentiable. As a result, for the rise and fall phases we have the following boundary conditions

$$\begin{aligned} s(0) &= 0, \quad s\left(\frac{2\pi}{3}\right) = 20, \quad s\left(\frac{4\pi}{3}\right) = 0, \\ v(0) &= v\left(\frac{4\pi}{3}\right) = 0, \\ a(0) &= a\left(\frac{4\pi}{3}\right) = 0, \end{aligned}$$

where $v(\theta) = s'(\theta)$, and $a(\theta) = s''(\theta)$. There are 7 boundary conditions to satisfy so we need the cam displacement polynomial to be at least degree 6. Thus, we set $s(\theta) = \sum_{k=0}^{6} \alpha_k \theta^k$, where α_k are coefficients to be solved for using the boundary conditions. We have

$$s'(\theta) = v(\theta) = \sum_{k=1}^{6} k\alpha_k\theta^{k-1}, \quad s''(\theta) = a(\theta) = \sum_{k=2}^{6} k(k-1)\alpha_k\theta^{k-2}.$$

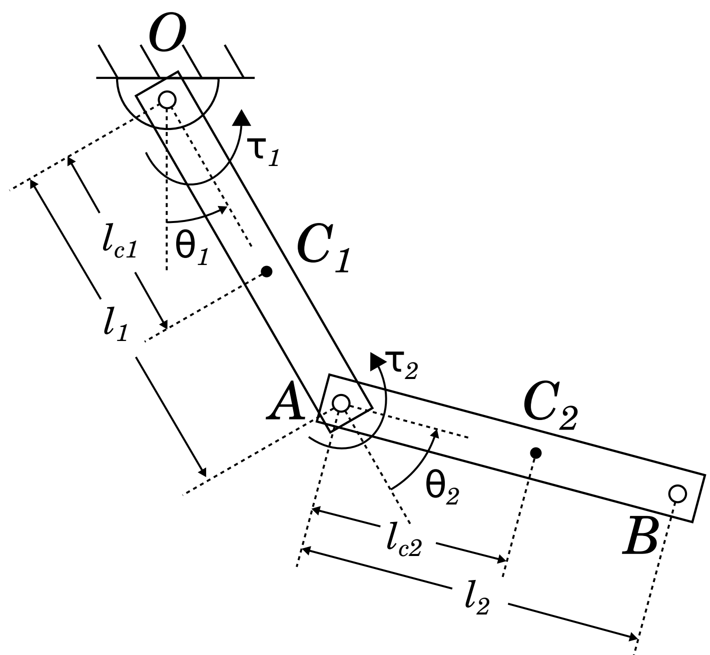
Plugging in the boundary conditions gives the following linear equation for the coefficients α_k:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{2\pi}{3} & \left(\frac{2\pi}{3}\right)^2 & \left(\frac{2\pi}{3}\right)^3 & \left(\frac{2\pi}{3}\right)^4 & \left(\frac{2\pi}{3}\right)^5 & \left(\frac{2\pi}{3}\right)^6 \\ 1 & \frac{4\pi}{3} & \left(\frac{4\pi}{3}\right)^2 & \left(\frac{4\pi}{3}\right)^3 & \left(\frac{4\pi}{3}\right)^4 & \left(\frac{4\pi}{3}\right)^5 & \left(\frac{4\pi}{3}\right)^6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\left(\frac{4\pi}{3}\right) & 3\left(\frac{4\pi}{3}\right)^2 & 4\left(\frac{4\pi}{3}\right)^3 & 5\left(\frac{4\pi}{3}\right)^4 & 6\left(\frac{4\pi}{3}\right)^5 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6\left(\frac{4\pi}{3}\right) & 12\left(\frac{4\pi}{3}\right)^2 & 20\left(\frac{4\pi}{3}\right)^3 & 30\left(\frac{4\pi}{3}\right)^4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Question 5 .... *0 (bonus) points*

Consider the planar double pendulum mechanism shown in the figure below. Points C_1 and C_2 are the centers of mass of link 1 and 2, respectively. The relevant link lengths are provided in the table below, on the left. The table also contains the masses and moments of inertia of the links, along the axis perpendicular to the plane about the centers of mass.

Physical Quantity	Values
l_1	0.5 m
l_2	0.5 m
l_{c1}	0.35 m
l_{c2}	0.3 m
m_1	0.5 kg
m_2	0.4 kg
$I_{1,33}$	$0.025\ \mathrm{kg}\cdot\mathrm{m}^2$
$I_{2,33}$	$0.02\ \mathrm{kg}\cdot\mathrm{m}^2$



In this problem, we will go through the derivation of `Lagrange's equations of motion` (Lagrangian dynamics) and compare the resulting equations with Newton-Euler equations of motion.

(a) Compute the kinetic energy of the double pendulum in terms of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. The kinetic energy of the system is given by its translational kinetic energy plus its rotational kinetic energy

$$\mathcal{K} = \mathcal{K}_{trans} + \mathcal{K}_{rot}.$$

The translational kinetic energy of the system is due to the masses and the linear velocities of the centers of mass of the links

$$\mathcal{K}_{trans} = \frac{1}{2} m_1 {}^0v^{C_1} \cdot {}^0v^{C_1} + \frac{1}{2} m_2 {}^0v^{C_2} \cdot {}^0v^{C_2},$$

while the rotational kinetic energy of the system is due to the moments of inertia and the angular velocities of the links

$$\mathcal{K}_{rot} = \frac{1}{2} {}^0\omega^1 \cdot I_1 {}^0\omega^1 + \frac{1}{2} {}^0\omega^2 \cdot I_2 {}^0\omega^2.$$

Solution: The position vectors from O to C_1 and C_2 are given by

$$\begin{aligned} p^{OC_1} &= l_{c1}\begin{bmatrix}\sin\theta_1 & -\cos\theta_1 & 0\end{bmatrix}^\top, \\ p^{OC_2} &= l_1\begin{bmatrix}\sin\theta_1 & -\cos\theta_1 & 0\end{bmatrix}^\top + l_{c2}\begin{bmatrix}\sin(\theta_1+\theta_2) & -\cos(\theta_1+\theta_2) & 0\end{bmatrix}^\top. \end{aligned}$$

Thus, the velocities are

$$\begin{aligned} {}^0v^{C_1} &= l_{c1}\dot\theta_1\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\end{bmatrix}^\top, \\ {}^0v^{C_2} &= l_1\dot\theta_1\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\end{bmatrix}^\top + l_{c2}(\dot\theta_1+\dot\theta_2)\begin{bmatrix}\cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0\end{bmatrix}^\top. \end{aligned}$$

The angular velocities of the links are given by

$$\begin{aligned} {}^0\omega^1 &= \dot\theta_1 z_0 \\ {}^0\omega^2 &= (\dot\theta_1+\dot\theta_2) z_0 \end{aligned}$$

Plugging these in the formulas for $\mathcal{K}_{trans}$ and $\mathcal{K}_{rot}$, we find the kinetic energy of the system to be

$$\begin{aligned} \mathcal{K} = &\frac{1}{2}\left(I_{1,33}+I_{2,33}+m_1l_{c1}^2+m_2\left(l_1^2+l_{c2}^2+2l_1l_{c2}\cos\theta_2\right)\right)\dot\theta_1^2 \\ &+\frac{1}{2}\left(I_{2,33}+m_2l_{c2}^2\right)\dot\theta_2^2+\left(I_{2,33}+m_2l_{c2}^2+m_2l_1l_{c2}\cos\theta_2\right)\dot\theta_1\dot\theta_2 \end{aligned}$$

(b) Compute the potential energy of the double pendulum in terms of θ_1 and θ_2. The potential energy of the system is due to earth's gravitational field, assumed to point downwards in the plane of motion. The elevation of the center of mass of a link is directly proportional to its potential energy; hence the total potential energy is given by

$$\mathcal{P} = m_1 g p^{OC_1}\cdot y_0 + m_2 g p^{OC_2}\cdot y_0,$$

where $g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2}$ is the constant of gravitational acceleration. p^{OC_1} and p^{OC_2} are the position vectors from O to the centers of mass of link 1 and link 2, respectively.

Solution: Using the position vector found in part (a) in the formula given above, we get

$$\mathcal{P} = -\left[(m_1l_{c1}+m_2l_1)\cos\theta_1 + m_2l_{c2}\cos(\theta_1+\theta_2)\right]g$$

(c) [30 points] Compute the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P},$$

in terms of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. Use this Lagrangian to compute Lagrange's equations of motion, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} = \tau_i, \quad i = 1, 2.$$

Here, τ_1 and τ_2 are the torques, applied to link 1 from the world, and to link 2 from link 1 by electrical motors that actuate the revolute joints at points O and A.

Solution: Lagrange's equations of motion are given by

$$\begin{aligned}
&\left(I_{1,33} + I_{2,33} + m_1 l_{c1}^2 + m_2\left(l_1^2 + l_{c2}^2\right) + 2m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_1 \\
&\quad + \left(I_{2,33} + m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_2 - m_2 l_1 l_{c2}\sin(\theta_2)\dot{\theta}_2\left(2\dot{\theta}_1 + \dot{\theta}_2\right) \\
&\quad + g\left((m_1 l_{c1} + m_2 l_1)\sin(\theta_1) + m_2 l_{c2}\sin(\theta_1 + \theta_2)\right) = \tau_1,
\end{aligned} \tag{9a}$$

$$\begin{aligned}
&\left(I_{2,33} + m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_1 + \left(I_{2,33} + m_2 l_{c2}^2\right)\ddot{\theta}_2 \\
&\quad + m_2 l_1 l_{c2}\sin(\theta_2)\dot{\theta}_1^2 + m_2 g l_{c2}\sin(\theta_1 + \theta_2) = \tau_2.
\end{aligned} \tag{9b}$$

(d) Compute the Newton-Euler equations of motion for the double pendulum. If at a particular instant $(\theta_1, \theta_2) = (\frac{\pi}{3}, \frac{\pi}{4})$ rad, $(\dot{\theta}_1, \dot{\theta}_2) = (0.1, -0.2)\ \frac{\text{rad}}{\text{s}}$, and $(\tau_1, \tau_2) = (0.5, -0.2)$ Nm., compute the accelerations $(\ddot{\theta}_1, \ddot{\theta}_2)$ using both Lagrange's and Newton-Euler equations of motion. Compare your results. Do you get the same result using either method? Explain why or why not.

Solution:

We compute the dynamic force/moment balance about the center of mass of each link. Newton-Euler equations of motion are given by

$$A\xi = b,$$

where

$$A = \begin{bmatrix}
1 & 0 & -1 & 0 & -m_1 l_{c1}\cos\theta_1 & 0 \\
0 & 1 & 0 & -1 & -m_1 l_{c1}\sin\theta_1 & 0 \\
0 & 0 & -l_1\cos\theta_1 & -l_1\sin\theta_1 & -I_{1,33} - m_1 l_{c1}^2 & 0 \\
0 & 0 & 1 & 0 & -m_2\left(l_1\cos\theta_1 + l_{c2}\cos(\theta_1+\theta_2)\right) & -m_2 l_{c2}\cos(\theta_1+\theta_2) \\
0 & 0 & 0 & 1 & -m_2\left(l_1\sin\theta_1 + l_{c2}\sin(\theta_1+\theta_2)\right) & -m_2 l_{c2}\sin(\theta_1+\theta_2) \\
0 & 0 & 0 & 0 & -I_{2,33} - m_2 l_{c2}^2 - m_2 l_1 l_{c2}\cos\theta_2 & -I_{2,33} - m_2 l_{c2}^2
\end{bmatrix},$$

$$\xi = \begin{bmatrix} f_{01,x} & f_{01,y} & f_{12,x} & f_{12,y} & \ddot{\theta}_1 & \ddot{\theta}_2 \end{bmatrix}^\top,$$

$$b = \begin{bmatrix} m_1 l_{c1} \sin(\theta_1)\dot{\theta}_1^2 \\ -m_1 \left(g + l_{c1}\cos(\theta_1)\dot{\theta}_1^2\right) \\ \tau_1 - \tau_2 - m_1 g l_{c1} \sin\theta_1 \\ m_2 \left(l_1 \sin(\theta_1)\dot{\theta}_1^2 + l_{c2}\sin(\theta_1+\theta_2)\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right) \\ -m_2 \left(g + l_1\cos(\theta_1)\dot{\theta}_1^2 + l_{c2}\cos(\theta_1+\theta_2)\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right) \\ \tau_2 - m_2 l_{c2}\left(g\sin(\theta_1+\theta_2) + l_1\sin(\theta_2)\dot{\theta}_1^2\right) \end{bmatrix}.$$

Using the constants given in the problem statement and the particular state of the mechanism, we get the solution $(\ddot{\theta}_1, \ddot{\theta}_2) = (-9.55276, -7.09408)\ \frac{\mathrm{rad}}{\mathrm{s}^2}$.

This result should be the same for both Lagrange's and Newton-Euler formulation since they are both accurate representations of the physical motion of the mechanism.