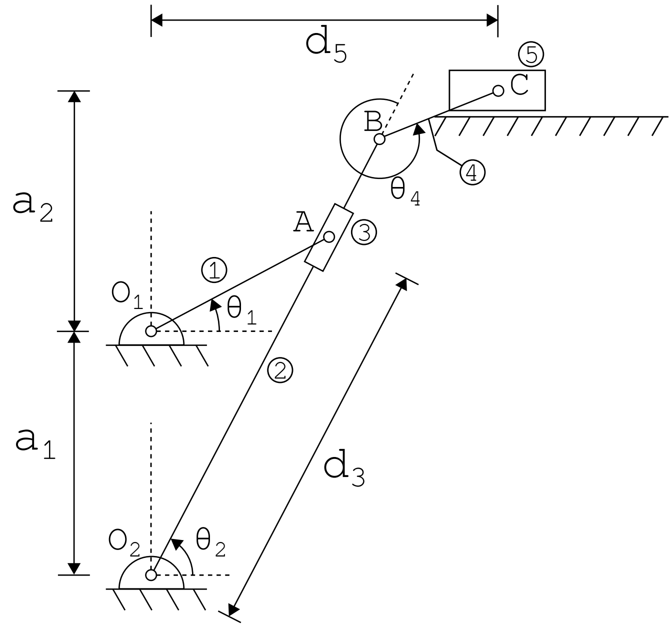
ME 380: Kinematics and Machine Dynamics
Final Examination Solutions

1. [30 points] Consider the Whitworth quick-return mechanism shown the figure below. The lengths of links 1, 2, and 4 are given by l_1, l_2, and l_4. The position of link 3 along link 2 is given by d_3 and the horizontal position of link 5 is given by d_5.

Quantity	Length [m]
a_1	1.0
a_2	1.0
l_1	$\frac{3}{4}$
l_2	$\frac{7}{4}$
l_4	$\frac{2}{3}$



(a) [10 points] Given θ_1, write down the loop equations used to solve for θ_2, d_3, θ_4, and d_5. Express the vector form of the equations in the world ($\Sigma_0 = \{x_0, y_0, z_0\}$) frame.

Solution: We need two vector loop equations in order to solve for 4 scalar variables. These loop equations are

$$\begin{aligned} p^{O_1A_1} + r - p^{O_2A_2} + a_1 y_0 &= 0, \\ p^{O_2B} + p^{BC} - d_5 x_0 - (a_1 + a_2) y_0 &= 0. \end{aligned} \tag{1}$$

where $r = \overrightarrow{A_1A_2}$. Here A_1 and A_2 are points coincident with point A at the instant in consideration with point A_1 fixed on link 1 and A_2 fixed on link 2. Writing out these equations in the world frame Σ_0 gives

$$\begin{aligned} l_1 \cos(\theta_1) - d_3 \cos(\theta_2) &= 0, \\ l_1 \sin(\theta_1) - d_3 \sin(\theta_2) + a_1 &= 0, \\ l_2 \cos(\theta_2) + l_4 \cos(\theta_2 + \theta_4) - d_5 &= 0, \\ l_2 \sin(\theta_2) + l_4 \sin(\theta_2 + \theta_4) - (a_1 + a_2) &= 0. \end{aligned}$$

(b) [10 points] Given ${}^0\omega^1 = \dot{\theta}_1 z_0$, write down the equations to find ${}^0\omega^2 = \dot{\theta}_2 z_0$, $\dot{d}_3$, ${}^0\omega^4 = \dot{\theta}_4 z_0$, and $\dot{d}_5$ in matrix form.

Solution: Differentiating equation (1) in the 0^{th} frame gives

$$
\begin{aligned}
{}^0\omega^1 \times p^{O_1A_1} - {}^2v^{A_1} - {}^0\omega^2 \times r - {}^0\omega^2 \times p^{O_2A_2} &= 0, \\
{}^0\omega^2 \times p^{O_2B} + {}^0\omega^4 \times p^{BC} - \dot{d}_5 x_0 &= 0.
\end{aligned} \tag{2}
$$

Writing out these equations in the world frame Σ_0 gives $J\dot{q} = 0$, where $q = (\theta_1, \theta_2, d_3, \theta_4, d_5)$. Notice the block diagonal structure of the matrix J.

$$
\begin{bmatrix}
-l_1 \sin\theta_1 & d_3 \sin\theta_2 & -\cos\theta_2 & 0 & 0 \\
l_1 \cos\theta_1 & -d_3 \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\
0 & -l_2 \sin\theta_2 - l_4 \sin(\theta_2 + \theta_4) & 0 & -l_4 \sin(\theta_2 + \theta_4) & -1 \\
0 & l_2 \cos\theta_2 + l_4 \cos(\theta_2 + \theta_4) & 0 & l_4 \cos(\theta_2 + \theta_4) & 0
\end{bmatrix}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{d}_5 \end{bmatrix} = 0. \tag{3}
$$

(c) [10 points] Given ${}^0\alpha^1 = \ddot{\theta}_1 z_0$, write down the equations to find ${}^0\alpha^2 = \ddot{\theta}_2 z_0$, $\ddot{d}_3$, ${}^0\alpha^4 = \ddot{\theta}_4 z_0$, and $\ddot{d}_5$ (You can leave the equations in vector form, i.e., no need to write out the matrix form of the equations).

Solution: Differentiating equation (2) in the world frame Σ_0 gives

$$
\begin{aligned}
&{}^0\alpha^1 \times p^{O_1A_1} + {}^0\omega^1 \times \left({}^0\omega^1 \times p^{O_1A_1}\right) - {}^2a^{A_1} - {}^0\alpha^2 \times r - {}^0\omega^2 \\
&\quad \times \left({}^0\omega^2 \times r\right) - 2\,{}^0\omega^2 \times {}^2v^{A_1} - {}^0\alpha^2 \times p^{O_2A_2} - {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2A_2}\right) = 0, \\
&{}^0\alpha^2 \times p^{O_2B} + {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2B}\right) + {}^0\alpha^4 \times p^{BC} + {}^0\omega^4 \times \left({}^0\omega^4 \times p^{BC}\right) - \ddot{d}_5 x_0 = 0.
\end{aligned} \tag{4}
$$

(d) [10 points (bonus)] If, at this instant, the position-level kinematic analysis is performed with the solution $(\theta_1, \theta_2, d_3, \theta_4, d_5) = (30°, 64.715°, 1.521, -25.923°, 1.267)$, and $\dot{\theta}_1 = -\frac{1}{2}\,\frac{\text{rad}}{\text{s}}$, find $\dot{d}_5$.

Solution: Plugging given values in equation (3) and solving the linear equations, we get the solution

$$
\begin{bmatrix} \dot{\theta}_2 & \dot{d}_3 & \dot{\theta}_4 & \dot{d}_5 \end{bmatrix} = \begin{bmatrix} -0.2027 & -0.2136 & 0.4943 & 0.1990 \end{bmatrix},
$$

where the units of the angular and linear velocities are radians per second and meters per second, respectively.

2. [40 points] Consider the toggle linkage shown in Figure 1. The masses of all of the links are uniformly distributed. Gravity acts in the downward direction within the plane of operation. Assume that the center of mass of link 5 is at the point B. In your solutions, make sure to use the numbering and naming convention given in Figure 1.

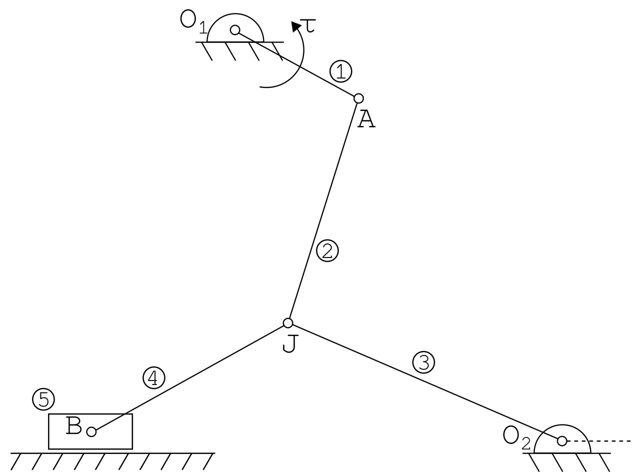


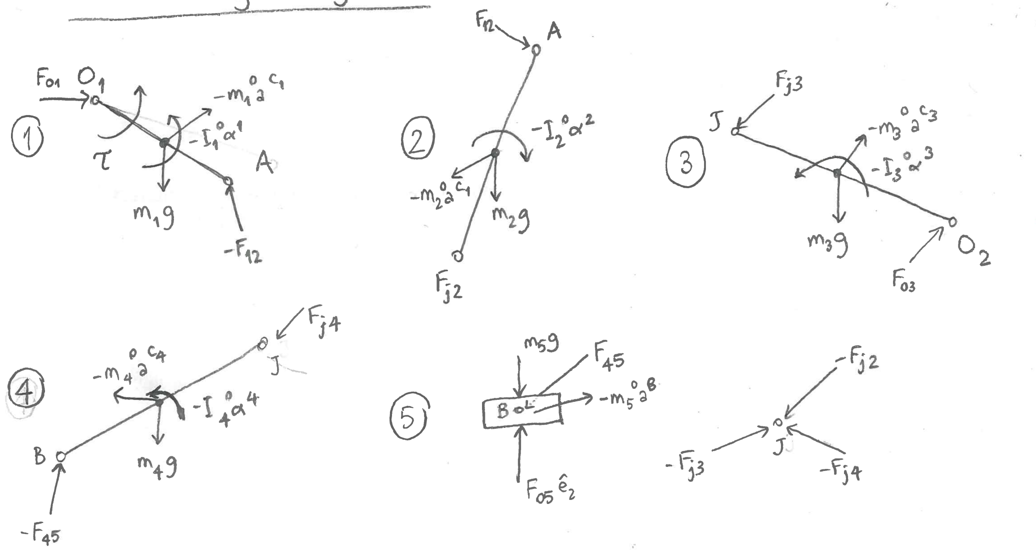
Figure 1: `Toggle Linkage`

(a) [17 points] Draw the free body diagram of each rigid body, marking all the external forces and moments as well as the inertial forces and moments. (Hint: To simplify the free-body diagrams at the multiple link junction, consider the point J as an auxiliary rigid body with no mass.)

(b) [23 points] *Forward dynamics* is the problem of finding the link accelerations ${}^0\alpha^k,\ k = 1, \ldots, 4$ and ${}^0a^B$, given the input torque, τ. Derive the Newton-Euler equations for forward dynamics in vector form. There is **no need** to write out the matrix form of the equations. Clearly state what the unknowns are and how many of them there are. Count the number of equations you have derived to solve for these unknowns.

Solution:

(a) Free - Body Diagrams



(b) Newton-Euler Forward Dynamics.
or
(c)

Dynamic Force Balance

$$
\begin{cases}
F_{01} - F_{12} - m_1 g \hat{e}_2 - m_1 {}^{o}a^{c_1} = 0 \\
F_{12} - F_{j2} - m_2 g \hat{e}_2 - m_2 {}^{o}a^{c_2} = 0 \\
F_{j3} + F_{03} - m_3 g \hat{e}_2 - m_3 {}^{o}a^{c_3} = 0 \\
F_{j4} - F_{45} - m_4 g \hat{e}_2 - m_4 {}^{o}a^{c_4} = 0 \\
F_{45} - F_{05}\hat{e}_2 - m_5 g \hat{e}_2 - m_5 {}^{o}a^{B} = 0 \\
F_{j2} + F_{j3} + F_{j4} = 0
\end{cases}
$$

12 eqns.

Dynamic Moment Balance

$$
\begin{cases}
p^{c_1 O_1} \times F_{01} - p^{c_1 A} \times F_{12} + \tau - I_1 {}^{o}\alpha^1 = 0 \\
p^{c_2 A} \times F_{12} + p^{c_2 J} \times F_{j2} - I_2 {}^{o}\alpha^2 = 0 \\
p^{c_3 J} \times F_{j3} + p^{c_3 O_2} \times F_{03} - I_3 {}^{o}\alpha^3 = 0 \\
p^{c_4 J} \times F_{j4} - p^{c_4 B} \times F_{45} - I_4 {}^{o}\alpha^4 = 0
\end{cases}
$$

4 eqns.

Let $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AJ}$, $p_3 = \overrightarrow{JO_2}$, $p_4 = \overrightarrow{JB}$, $p_5 = \overrightarrow{O_2B}$.

Acceleration Level Kinematics

$$
\begin{cases}
\sum_{k=1}^{3} {}^{o}\alpha^k \times p_k + {}^{o}\omega^k \times ({}^{o}\omega^k \times p_k) = 0 \\
-{}^{o}\alpha^3 \times p_3 - {}^{o}\omega^3 \times ({}^{o}\omega^3 \times p_3) + {}^{o}\alpha^4 \times p_4 + {}^{o}\omega^4 \times ({}^{o}\omega^4 \times p_4) - {}^{o}a^B = 0
\end{cases}
$$

4 eqns.

$12 + 4 + 4 = \underline{\underline{20 \text{ eqns.}}}$

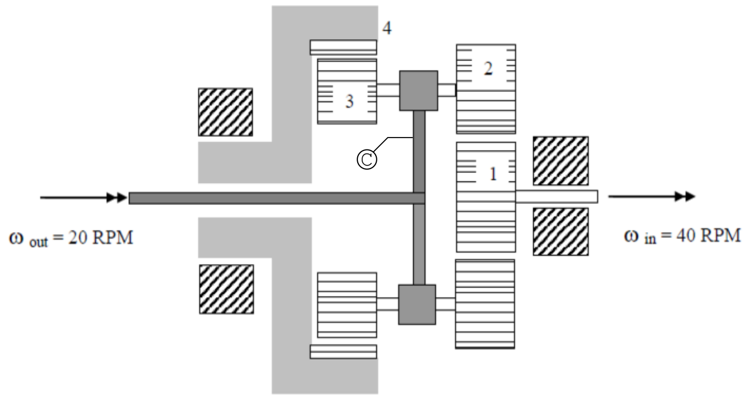
Unknowns

$F_{01}, F_{12}, F_{j2}, F_{j3}, F_{03}, F_{j4}, F_{45}, F_{05}, {}^{o}\alpha^1, {}^{o}\alpha^2, {}^{o}\alpha^3, {}^{o}\alpha^4, {}^{o}a^B.$

$2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$

$= \underline{\underline{20 \text{ unknowns.}}}$

3. [25 points] The ring gear in the epicyclic geartrain shown below is *not* fixed to the ground. The tooth numbers are $N_1 = 27$, $N_2 = 36$, $N_3 = 24$, $N_4 = 87$. Find the angular velocity of the ring gear. (Hint: Gears 2 and 3 are parts of the same rigid body.)



Solution: Let ${}^0\omega^C = \omega_C z_0$. We are given that $\omega_C = 20\,\text{rpm}$ and $\omega_1 = 40\,\text{rpm}$. Let P_1 and P_2 be the pitch point(s) between gears 1 and 2 with P_1 fixed on gear 1 and P_2 fixed on gear 2. We have

$$ {}^0v^{P_1} = \omega_1 r_1 y_0 = (\omega_C(r_1 + r_2) - \omega_2 r_2)\, y_0 =^0 v^{P_2}. \tag{5}$$

Let Q_3 and Q_4 be the pitch point(s) between gears 3 and 4 with Q_3 fixed on gear 3 and Q_4 fixed on gear 4. We have

$$ {}^0v^{Q_3} = (\omega_C(r_1 + r_2) + \omega_3 r_3)\, y_0 = \omega_4 r_4 y_0 =^0 v^{Q_4}. \tag{6}$$

Additionally, we know that $\omega_2 = \omega_3$ since these gears are physically mounted on the same rigid body. Solving equation (5) for $\omega_2 r_3$, we find

$$\omega_2 r_3 = \omega_C \frac{(r_1 + r_2) r_3}{r_2} - \omega_1 \frac{r_1}{r_2},$$

Plugging this back in equation (6) as well as $\omega_3 = \omega_2$, we have

$$\omega_4 r_4 = \omega_C \left(r_1 + r_2 + r_3 + \frac{r_1 r_3}{r_2} \right) - \omega_1 \frac{r_1 r_3}{r_2},$$

Now, notice that $r_4 = r_1 + r_2 + r_3$, so the above expression simplifies to

$$\omega_4 = \omega_C \left(1 + \frac{r_1}{r_2}\frac{r_3}{r_4} \right) - \omega_1 \frac{r_1}{r_2}\frac{r_3}{r_4} = \omega_C \left(1 + \frac{N_1}{N_2}\frac{N_3}{N_4} \right) - \omega_1 \frac{N_1}{N_2}\frac{N_3}{N_4}.$$

Plugging in the numerical values for N_1, N_2, N_3, N_4, ω_C and ω_1 yields
$\boxed{\omega_4 = 15.862\,\text{rpm}}$.

4. [15 points] A CEP (critical extreme position) cam is being designed to provide the following motion to a translating follower

Follower Displacement		Cam Rotation [rad]
Rise	0 cm to 20 cm	0 rad to $\frac{2\pi}{3}$ rad
Fall	20 cm to 0 cm	$\frac{2\pi}{3}$ rad to $\frac{4\pi}{3}$ rad
Dwell	at 0 cm	$\frac{4\pi}{3}$ rad to 2π rad

Find a matrix representations for the coefficients of a **single** polynomial displacement function for the rise and fall portion of the follower. Clearly state your boundary conditions at the beginning of your solution.

Solution: The fundamental law of cam design states that the cam displacement function must at least be twice differentiable. As a result, for the rise and fall phases we have the following boundary conditions

$$\begin{aligned} s(0) &= 0, \quad s\left(\frac{2\pi}{3}\right) = 20, \quad s\left(\frac{4\pi}{3}\right) = 0, \\ v(0) &= v\left(\frac{4\pi}{3}\right) = 0, \\ a(0) &= a\left(\frac{4\pi}{3}\right) = 0, \end{aligned}$$

where $v(\theta) = s'(\theta)$, and $a(\theta) = s''(\theta)$. There are 7 boundary conditions to satisfy so we need the cam displacement polynomial to be at least degree 6. Thus, we set $s(\theta) = \sum_{k=0}^{6} \alpha_k \theta^k$, where α_k are coefficients to be solved for using the boundary conditions. We have

$$s'(\theta) = v(\theta) = \sum_{k=1}^{6} k\alpha_k \theta^{k-1}, \quad s''(\theta) = a(\theta) = \sum_{k=2}^{6} k(k-1)\alpha_k \theta^{k-2}.$$

Plugging in the boundary conditions gives the following linear equation for the coefficients α_k:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{2\pi}{3} & \left(\frac{2\pi}{3}\right)^2 & \left(\frac{2\pi}{3}\right)^3 & \left(\frac{2\pi}{3}\right)^4 & \left(\frac{2\pi}{3}\right)^5 & \left(\frac{2\pi}{3}\right)^6 \\ 1 & \frac{4\pi}{3} & \left(\frac{4\pi}{3}\right)^2 & \left(\frac{4\pi}{3}\right)^3 & \left(\frac{4\pi}{3}\right)^4 & \left(\frac{4\pi}{3}\right)^5 & \left(\frac{4\pi}{3}\right)^6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\left(\frac{4\pi}{3}\right) & 3\left(\frac{4\pi}{3}\right)^2 & 4\left(\frac{4\pi}{3}\right)^3 & 5\left(\frac{4\pi}{3}\right)^4 & 6\left(\frac{4\pi}{3}\right)^5 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6\left(\frac{4\pi}{3}\right) & 12\left(\frac{4\pi}{3}\right)^2 & 20\left(\frac{4\pi}{3}\right)^3 & 30\left(\frac{4\pi}{3}\right)^4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$