ME 380

Kinematics and Machine Dynamics
Final Exam Solutions

1. [30 points] Consider the Whitworth quick-return mechanism shown in Figure 1. The lengths of links 1, 2, and 4 are given by l_1, l_2, and l_4. The position of link 3 along link 2 is given by s_3 and the horizontal position of link 5 is given by s_5.

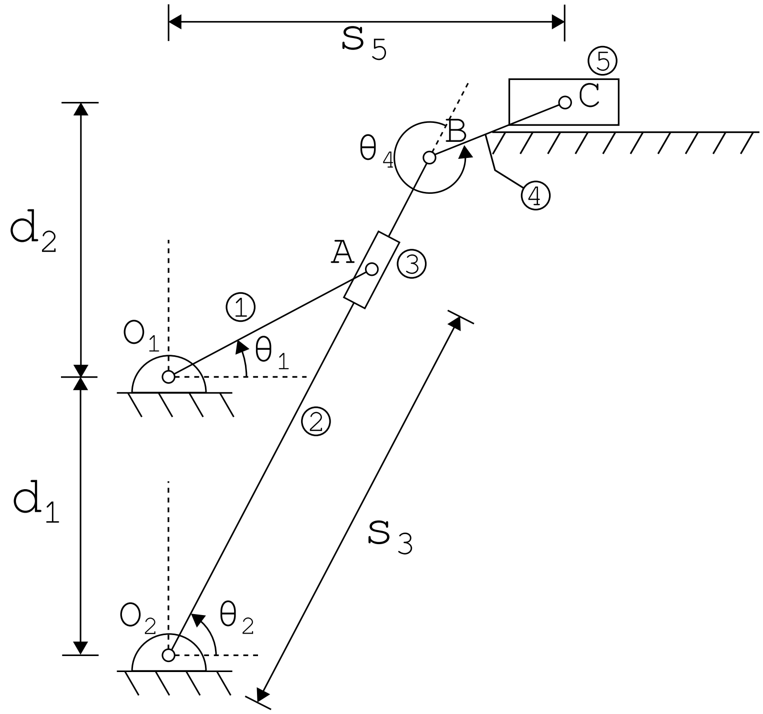


Figure 1: `Whitworth quick-return mechanism`

(a) [10 points] Given θ_1, write down the loop equations from which one can solve for θ_2, s_3, θ_4, and s_5.

Solution: We need two vector loop equations in order to solve for 4 scalar variables. These loop equations are

$$\begin{aligned} p^{O_1A_1} + r - p^{O_2A_2} + d_1\hat{e}_2 &= 0, \\ p^{O_2B} + p^{BC} - s_5\hat{e}_1 - (d_1 + d_2)\hat{e}_2 &= 0. \end{aligned} \tag{1}$$

where $r = \overrightarrow{A_1A_2}$. Here A_1 and A_2 are points coincident with point A at the instant in consideration with point A_1 fixed on link 1 and A_2 fixed on link 2. Writing out these equations in the frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ gives

$$
\begin{aligned}
l_1 \cos(\theta_1) - s_3 \cos(\theta_2) &= 0, \\
l_1 \sin(\theta_1) - s_3 \sin(\theta_2) + d_1 &= 0, \\
l_2 \cos(\theta_2) + l_4 \cos(\theta_2 + \theta_4) - s_5 &= 0, \\
l_2 \sin(\theta_2) + l_4 \sin(\theta_2 + \theta_4) - (d_1 + d_2) &= 0.
\end{aligned}
$$

(b) [10 points] Given ${}^0\omega^1 = \dot{\theta}_1 \hat{e}_3$, write down the equations to find ${}^0\omega^2$, $\dot{s}_3$, ${}^0\omega^4$, and $\dot{s}_5$ in matrix form.

Solution: Differentiating equation (1) in the 0^{th} frame gives

(2)
$$
\begin{aligned}
{}^0\omega^1 \times p^{O_1A_1} -{}^2 v^{A_1} - {}^0\omega^2 \times r -{}^0 \omega^2 \times p^{O_2A_2} &= 0, \\
{}^0\omega^2 \times p^{O_2B} + {}^0\omega^4 \times p^{BC} - \dot{s}_5 \hat{e}_1 &= 0.
\end{aligned}
$$

Writing out these equations in the frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ gives

$$
\begin{aligned}
-l_1\dot{\theta}_1 \sin(\theta_1) - \dot{s}_3 \cos(\theta_2) + s_3 \dot{\theta}_2 \sin\theta_2 &= 0, \\
l_1\dot{\theta}_1 \cos(\theta_1) - \dot{s}_3 \sin(\theta_2) - s_3 \dot{\theta}_2 \cos(\theta_2) &= 0, \\
-l_2\dot{\theta}_2 \sin(\theta_2) - l_4 \left(\dot{\theta}_2 + \dot{\theta}_4\right) \sin(\theta_2 + \theta_4) - \dot{s}_5 &= 0, \\
l_2\dot{\theta}_2 \cos(\theta_2) + l_4 \left(\dot{\theta}_2 + \dot{\theta}_4\right) \cos(\theta_2 + \theta_4) &= 0.
\end{aligned}
$$

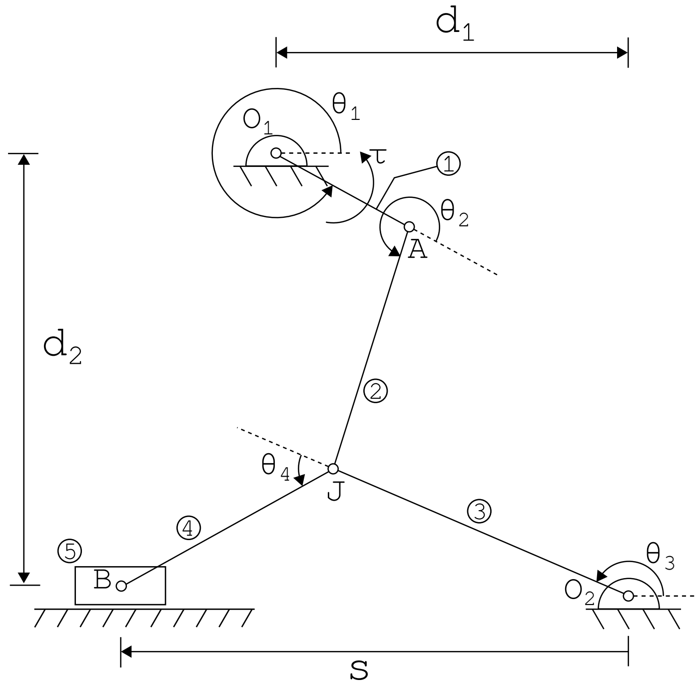
(c) [10 points] Given ${}^0\alpha^1 = \ddot{\theta}_1 \hat{e}_3$, write down the equations to find ${}^0\alpha^2$, $\ddot{s}_3$, ${}^0\alpha^4$, and $\ddot{s}_5$ (You can leave the equations in vector form, i.e., no need to write out the matrix form of the equations).

Solution: Differentiating equation (3) in the 0^{th} frame gives

(3)
$$
\begin{aligned}
&{}^0\alpha^1 \times p^{O_1A_1} + {}^0\omega^1 \times \left({}^0\omega^1 \times p^{O_1A_1}\right) - {}^2a^{A_1} - {}^0\alpha^2 \times r - {}^0\omega^2 \\
&\quad \times \left({}^0\omega^2 \times r\right) - 2\,{}^0\omega^2 \times {}^2v^{A_1} - {}^0\alpha^2 \times p^{O_2A_2} - {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2A_2}\right) = 0, \\
&{}^0\alpha^2 \times p^{O_2B} + {}^0\omega^2 \times \left({}^0\omega^2 \times p^{O_2B}\right) +{}^0 \alpha^4 \times p^{BC} + {}^0\omega^4 \times \left({}^0\omega^4 \times p^{BC}\right) - \ddot{s}_5 \hat{e}_1 = 0.
\end{aligned}
$$

2. [35 points] Consider the **toggle linkage** shown in the figure below. Links 1 through 4 have lengths $l_1, \ldots, l_4$, and their masses are uniformly distributed. Gravity acts in the downward direction within the plane of operation. Assume that the center of mass of link 5 is at the point B and that all the links have unit masses. The moment of inertia of the all links equals 10^{-3}. The values of the link lengths are defined in the table below.

Quantity	Length [units]
l_1	1.3
l_2	2.5
l_3	3.2
l_4	2.4
d_1	3.5
d_2	4.2



(a) [10 points] Draw the free-body diagram of each rigid body, marking all the external forces and moments as well as the inertial forces and moments. (Hint: To simplify the free-body diagrams at the multiple link junction, consider the point J as an auxiliary rigid body with no mass.)

(b) [10 points] Derive the matrix form of the equations for static force balance.

(c) [10 points] Derive the matrix form of the Newton-Euler equations for forward dynamics. (Forward dynamics is the problem of finding the link accelerations ${}^0\alpha^k,\ k = 1, \ldots, 4$ and ${}^0a^B$, given the input torque, τ.)

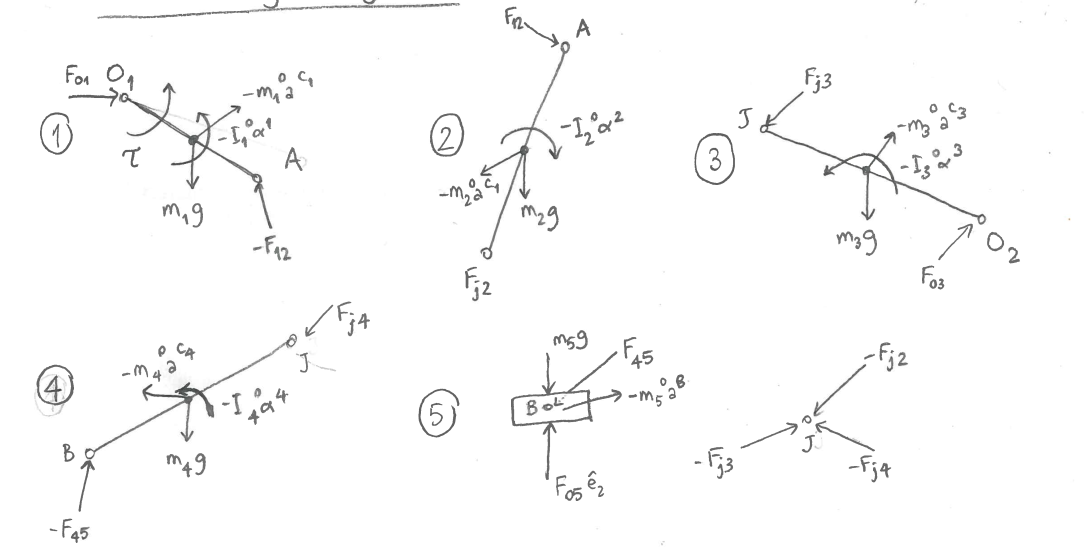
(d) [5 points] Assume that the kinematics problem has been solved as follows

$(\theta_1, \dot{\theta}_1, \ddot{\theta}_1)$	$\left(-\frac{\pi}{6}, -\frac{1}{2}, 0\right)$
$(\theta_2, \dot{\theta}_2, \ddot{\theta}_2)$	$(-1.298, 0.543, 0.179)$
$(\theta_3, \dot{\theta}_3, \ddot{\theta}_3)$	$(2.781, 0.197, -0.033)$
$(\theta_4, \dot{\theta}_4, \ddot{\theta}_4)$	$(0.850, -0.475, 0.101)$
$(s, \dot{s}, \ddot{s})$	$(-5.113, -0.536, -0.394)$

If at this instant $F_{01} = \begin{bmatrix}3.282 & 28.791\end{bmatrix}^\top$ and $F_{03} = \begin{bmatrix}-3.081 & 6.094\end{bmatrix}^\top$, find F_{05}.

Solution:

(a) Free - Body Diagrams



(b) Newton-Euler Forward Dynamics.
or
(c)

Dynamic Force Balance

$$F_{01} - F_{12} - m_1 g\hat{e}_2 - m_1\,{}^{0}a^{c_1} = 0$$
$$F_{12} - F_{j2} - m_2 g\hat{e}_2 - m_2\,{}^{0}a^{c_2} = 0$$
$$F_{j3} + F_{03} - m_3 g\hat{e}_2 - m_3\,{}^{0}a^{c_3} = 0$$
$$F_{j4} - F_{45} - m_4 g\hat{e}_2 - m_4\,{}^{0}a^{c_4} = 0$$
$$F_{45} - F_{05}\hat{e}_2 - m_5 g\hat{e}_2 - m_5\,{}^{0}a^{B} = 0$$
$$F_{j2} + F_{j3} + F_{j4} = 0$$

12 eqns.

Dynamic Moment Balance

$$p^{C_1O_1} \times F_{01} - p^{C_1A} \times F_{12} + \tau - I_1\,{}^{0}\alpha^{1} = 0$$
$$p^{C_2A} \times F_{12} + p^{C_1J} \times F_{j2} - I_2\,{}^{0}\alpha^{2} = 0$$
$$p^{C_3J} \times F_{j3} + p^{C_3O_2} \times F_{03} - I_3\,{}^{0}\alpha^{3} = 0$$
$$p^{C_4J} \times F_{j4} - p^{C_4B} \times F_{45} - I_4\,{}^{0}\alpha^{4} = 0$$

4 eqns.

Let $p_1 = \overrightarrow{O_1A}$, $p_2 = \overrightarrow{AJ}$, $p_3 = \overrightarrow{JO_2}$, $p_4 = \overrightarrow{JB}$, $p_5 = \overrightarrow{O_2B}$.

Acceleration Level Kinematics

$$\sum_{k=1}^{3} {}^{0}\alpha^{k} \times p_k + {}^{0}\omega^{k} \times ({}^{0}\omega^{k} \times p_k) = 0$$
$$-{}^{0}\alpha^{3} \times p_3 - {}^{0}\omega^{3} \times ({}^{0}\omega^{3} \times p_3) + {}^{0}\alpha^{4} \times p_4 + {}^{0}\omega^{4} \times ({}^{0}\omega^{4} \times p_4) - {}^{0}a^{B} = 0$$

4 eqns.

$12 + 4 + 4 = \underline{20 \text{ eqns.}}$

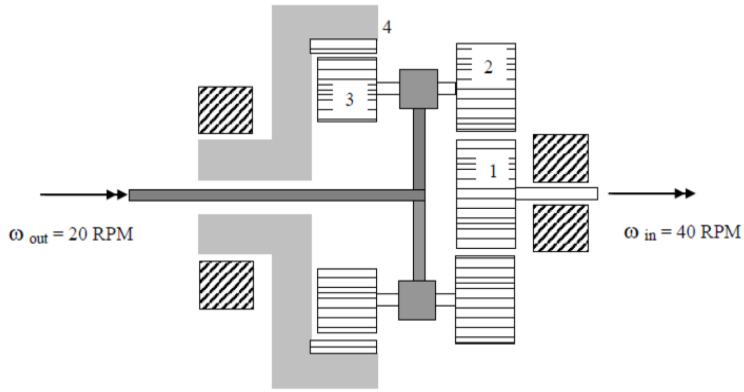
Unknowns

$F_{01}, F_{12}, F_{j2}, F_{j3}, F_{03}, F_{j4}, F_{45}, F_{05}, {}^{0}\alpha^{1}, {}^{0}\alpha^{2}, {}^{0}\alpha^{3}, {}^{0}\alpha^{4}, {}^{0}a^{B}.$

$2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$

$= \underline{20 \text{ unknowns.}}$

3. [25 points] The ring gear in the epicyclic geartrain shown below is *not* fixed to the ground. The tooth numbers are $N_1 = 17$, $N_2 = 28$, $N_3 = 15$, $N_4 = 60$. Find the angular velocity of the ring gear.



Solution: Let $\hat{e}_3$ denote the unit vector pointing out of the plane. Let C denote the carrier and let ${}^0\omega^C = \omega_C \hat{e}_3$. We are given that $\omega_C = 20\,\text{rpm}$, $\omega_1 = 40\,\text{rpm}$.

Let P_1 and P_2 be the pitch point(s) between gears 1 and 2 with P_1 fixed on gear 1 and P_2 fixed on gear 2. We have

$$ {}^0v^{P_1} = \omega_1 r_1 \hat{e}_1 = \left(\omega_C(r_1 + r_2) - \omega_2 r_2\right) \hat{e}_1 = {}^0\, v^{P_2}. \tag{4} $$

Let Q_3 and Q_4 be the pitch point(s) between gears 3 and 4 with Q_3 fixed on gear 3 and Q_4 fixed on gear 4. We have

$$ {}^0v^{Q_3} = \left(\omega_C(r_1 + r_2) + \omega_3 r_3\right) \hat{e}_1 = \omega_4 r_4 \hat{e}_1 = {}^0\, v^{Q_4}. \tag{5} $$

Additionally, we know that $\omega_2 = \omega_3$ since these gears are physically mounted on the same rigid body. Solving equation (4) for $\omega_2 r_3$, we find

$$ \omega_2 r_3 = \omega_C \frac{(r_1 + r_2) r_3}{r_2} - \omega_1 \frac{r_1}{r_2}, $$

Plugging this back in equation (5) as well as $\omega_3 = \omega_2$, we have

$$ \omega_4 r_4 = \omega_C \left(r_1 + r_2 + r_3 + \frac{r_1 r_3}{r_2} \right) - \omega_1 \frac{r_1 r_3}{r_2}, $$

Now, notice that $r_4 = r_1 + r_2 + r_3$, so the above expression simplifies to

$$ \omega_4 = \omega_C \left(1 + \frac{r_1}{r_2}\frac{r_3}{r_4} \right) - \omega_1 \frac{r_1}{r_2}\frac{r_3}{r_4} = \omega_C \left(1 + \frac{N_1}{N_2}\frac{N_3}{N_4} \right) - \omega_1 \frac{N_1}{N_2}\frac{N_3}{N_4}. $$

Plugging in the numerical values for N_1, N_2, N_3, N_4, ω_C and ω_1 yields $\boxed{\omega_4 = 16.964\,\text{rpm}}$.

4. [15 points] A CEP (critical extreme position) cam is being designed to provide the following motion to a translating follower

Follower Displacement		Cam Rotation [rad]
Rise	0 cm to 5 cm	0 rad to $\frac{\pi}{2}$ rad
Fall	5 cm to 0 cm	$\frac{\pi}{2}$ rad to π rad
Dwell	at 0 cm	π rad to 2π rad

Find a matrix representations for the coefficients of a **single** polynomial displacement function for the rise and fall portion of the follower. Clearly state your boundary conditions at the beginning of your solution.

Solution: The fundamental law of cam design states that the cam displacement function must at least be twice differentiable. As a result, for the rise and fall phases we have the following boundary conditions

$$\begin{aligned} s(0) &= 0, \quad s\left(\frac{\pi}{2}\right) = 5, \quad s(\pi) = 0, \\ v(0) &= v(\pi) = 0, \\ a(0) &= a(\pi) = 0, \end{aligned}$$

where $v(\theta) = s'(\theta)$, and $a(\theta) = s''(\theta)$. There are 7 boundary conditions to satisfy so we need the cam displacement polynomial to be at least degree 6. Thus, we set $s(\theta) = \sum_{k=0}^{6} \alpha_k \theta^k$, where α_k are coefficients to be solved for using the boundary conditions. We have

$$\begin{aligned} s'(\theta) &= v(\theta) = \sum_{k=1}^{6} k\alpha_k \theta^{k-1}, \\ s''(\theta) &= a(\theta) = \sum_{k=2}^{6} k(k-1)\alpha_k \theta^{k-2}. \end{aligned}$$

Plugging in the boundary conditions gives the following linear equation for the coefficients α_k:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{\pi}{2} & \left(\frac{\pi}{2}\right)^2 & \left(\frac{\pi}{2}\right)^3 & \left(\frac{\pi}{2}\right)^4 & \left(\frac{\pi}{2}\right)^5 & \left(\frac{\pi}{2}\right)^6 \\ 1 & \pi & \pi^2 & \pi^3 & \pi^4 & \pi^5 & \pi^6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\pi & 3\pi^2 & 4\pi^3 & 5\pi^4 & 6\pi^5 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6\pi & 12\pi^2 & 20\pi^3 & 30\pi^4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$