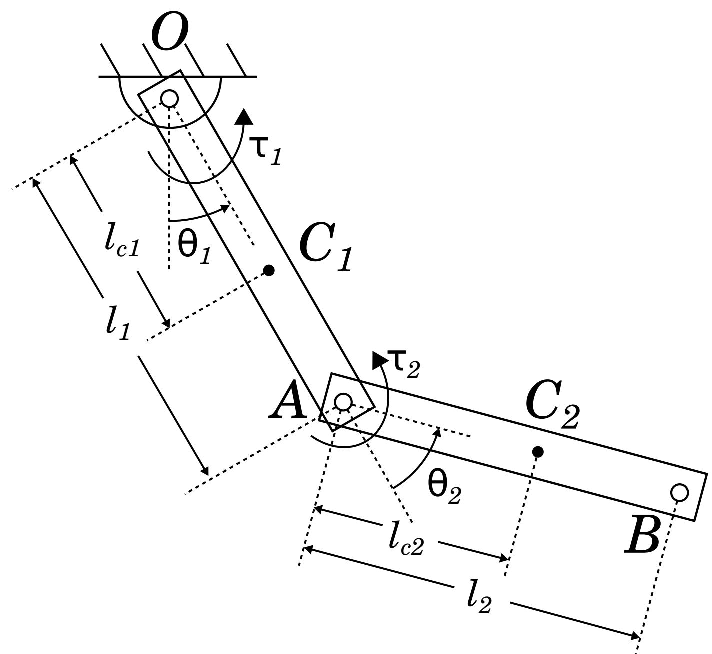
ME 380: Kinematics and Machine Dynamics
Final Examination Solutions

1. [100 points (bonus)] Consider the planar double pendulum mechanism shown in the figure below. Points C_1 and C_2 are the centers of mass of link 1 and 2, respectively. The relevant link lengths are provided in the table below, on the left. The table also contains the masses and moments of inertia of the links, along the axis perpendicular to the plane about the centers of mass.

Physical Quantity	Values
l_1	0.5 m
l_2	0.5 m
l_{c1}	0.35 m
l_{c2}	0.3 m
m_1	0.5 kg
m_2	0.4 kg
$I_{1,33}$	$0.025\ \mathrm{kg \cdot m^2}$
$I_{2,33}$	$0.02\ \mathrm{kg \cdot m^2}$



In this problem, we will go through the derivation of `Lagrange's equations of motion` (Lagrangian dynamics) and compare the resulting equations with Newton-Euler equations of motion.

(a) [25 points] Compute the kinetic energy of the double pendulum in terms of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. The kinetic energy of the system is given by its translational kinetic energy plus its rotational kinetic energy

$$\mathcal{K} = \mathcal{K}_{trans} + \mathcal{K}_{rot}.$$

The translational kinetic energy of the system is due to the masses and the linear velocities of the centers of mass of the links

$$\mathcal{K}_{trans} = \frac{1}{2} m_1 {}^0v^{C_1} \cdot {}^0v^{C_1} + \frac{1}{2} m_2 {}^0v^{C_2} \cdot {}^0v^{C_2},$$

while the rotational kinetic energy of the system is due to the moments of inertia and the angular velocities of the links

$$\mathcal{K}_{rot} = \frac{1}{2} {}^0\omega^1 \cdot I_1 {}^0\omega^1 + \frac{1}{2} {}^0\omega^2 \cdot I_2 {}^0\omega^2.$$

Solution: The position vectors from O to C_1 and C_2 are given by

$$p^{OC_1} = l_{c1}\begin{bmatrix}\sin\theta_1 & -\cos\theta_1 & 0\end{bmatrix}^\top,$$
$$p^{OC_2} = l_1\begin{bmatrix}\sin\theta_1 & -\cos\theta_1 & 0\end{bmatrix}^\top + l_{c2}\begin{bmatrix}\sin(\theta_1+\theta_2) & -\cos(\theta_1+\theta_2) & 0\end{bmatrix}^\top.$$

Thus, the velocities are

$${}^0v^{C_1} = l_{c1}\dot\theta_1\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\end{bmatrix}^\top,$$
$${}^0v^{C_2} = l_1\dot\theta_1\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\end{bmatrix}^\top + l_{c2}(\dot\theta_1+\dot\theta_2)\begin{bmatrix}\cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0\end{bmatrix}^\top.$$

The angular velocities of the links are given by

$$\begin{aligned}{}^0\omega^1 &= \dot\theta_1 z_0\\ {}^0\omega^2 &= (\dot\theta_1+\dot\theta_2)z_0\end{aligned}$$

Plugging these in the formulas for $\mathcal{K}_{trans}$ and $\mathcal{K}_{rot}$, we find the kinetic energy of the system to be

$$\begin{aligned}\mathcal{K} = &\frac{1}{2}\left(I_{1,33}+I_{2,33}+m_1l_{c1}^2+m_2\left(l_1^2+l_{c2}^2+2l_1l_{c2}\cos\theta_2\right)\right)\dot\theta_1^2\\ &+\frac{1}{2}\left(I_{2,33}+m_2l_{c2}^2\right)\dot\theta_2^2+\left(I_{2,33}+m_2l_{c2}^2+m_2l_1l_{c2}\cos\theta_2\right)\dot\theta_1\dot\theta_2\end{aligned}$$

(b) [10 points] Compute the potential energy of the double pendulum in terms of θ_1 and θ_2. The potential energy of the system is due to earth's gravitational field, assumed to point downwards in the plane of motion. The elevation of the center of mass of a link is directly proportional to its potential energy; hence the total potential energy is given by

$$\mathcal{P} = m_1gp^{OC_1}\cdot y_0 + m_2gp^{OC_2}\cdot y_0,$$

where $g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2}$ is the constant of gravitational acceleration. p^{OC_1} and p^{OC_2} are the position vectors from O to the centers of mass of link 1 and link 2, respectively.

Solution: Using the position vector found in part (a) in the formula given above, we get

$$\mathcal{P} = -\left[(m_1l_{c1}+m_2l_1)\cos\theta_1 + m_2l_{c2}\cos(\theta_1+\theta_2)\right]g$$

(c) [30 points] Compute the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P},$$

in terms of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. Use this Lagrangian to compute Lagrange's equations of motion, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} = \tau_i, \quad i = 1, 2.$$

Here, τ_1 and τ_2 are the torques, applied to link 1 from the world, and to link 2 from link 1 by electrical motors that actuate the revolute joints at points O and A.

Solution: Lagrange's equations of motion are given by

$$\begin{aligned} &\left(I_{1,33} + I_{2,33} + m_1 l_{c1}^2 + m_2\left(l_1^2 + l_{c2}^2\right) + 2 m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_1 \\ &\quad + \left(I_{2,33} + m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_2 - m_2 l_1 l_{c2}\sin(\theta_2)\dot{\theta}_2\left(2\dot{\theta}_1 + \dot{\theta}_2\right) \\ &\quad + g\left((m_1 l_{c1} + m_2 l_1)\sin(\theta_1) + m_2 l_{c2}\sin(\theta_1 + \theta_2)\right) = \tau_1, \end{aligned} \tag{1a}$$

$$\begin{aligned} &\left(I_{2,33} + m_2 l_{c2}^2 + m_2 l_1 l_{c2}\cos\theta_2\right)\ddot{\theta}_1 + \left(I_{2,33} + m_2 l_{c2}^2\right)\ddot{\theta}_2 \\ &\quad + m_2 l_1 l_{c2}\sin(\theta_2)\dot{\theta}_1^2 + m_2 g l_{c2}\sin(\theta_1 + \theta_2) = \tau_2. \end{aligned} \tag{1b}$$

(d) [35 points] Compute the Newton-Euler equations of motion for the double pendulum. If at a particular instant $(\theta_1, \theta_2) = (\frac{\pi}{3}, \frac{\pi}{4})$ rad, $(\dot{\theta}_1, \dot{\theta}_2) = (0.1, -0.2)\ \frac{\text{rad}}{\text{s}}$, and $(\tau_1, \tau_2) = (0.5, -0.2)$ Nm., compute the accelerations $(\ddot{\theta}_1, \ddot{\theta}_2)$ using both Lagrange's and Newton-Euler equations of motion. Compare your results. Do you get the same result using either method? Explain why or why not.

Solution:

We compute the dynamic force/moment balance about the center of mass of each link. Newton-Euler equations of motion are given by

$$A\xi = b,$$

where

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & -m_1 l_{c1} \cos\theta_1 & 0 \\ 0 & 1 & 0 & -1 & -m_1 l_{c1} \sin\theta_1 & 0 \\ 0 & 0 & -l_1 \cos\theta_1 & -l_1 \sin\theta_1 & -I_{1,33} - m_1 l_{c1}^2 & 0 \\ 0 & 0 & 1 & 0 & -m_2 \left(l_1 \cos\theta_1 + l_{c2} \cos\left(\theta_1 + \theta_2\right)\right) & -m_2 l_{c2} \cos\left(\theta_1 + \theta_2\right) \\ 0 & 0 & 0 & 1 & -m_2 \left(l_1 \sin\theta_1 + l_{c2} \sin\left(\theta_1 + \theta_2\right)\right) & -m_2 l_{c2} \sin\left(\theta_1 + \theta_2\right) \\ 0 & 0 & 0 & 0 & -I_{2,33} - m_2 l_{c2}^2 - m_2 l_1 l_{c2} \cos\theta_2 & -I_{2,33} - m_2 l_{c2}^2 \end{bmatrix},$$

$$\xi = \begin{bmatrix} f_{01,x} & f_{01,y} & f_{12,x} & f_{12,y} & \ddot{\theta}_1 & \ddot{\theta}_2 \end{bmatrix}^\top,$$

$$b = \begin{bmatrix} m_1 l_{c1} \sin\left(\theta_1\right) \dot{\theta}_1^2 \\ -m_1 \left(g + l_{c1} \cos\left(\theta_1\right) \dot{\theta}_1^2\right) \\ \tau_1 - \tau_2 - m_1 g l_{c1} \sin\theta_1 \\ m_2 \left(l_1 \sin\left(\theta_1\right) \dot{\theta}_1^2 + l_{c2} \sin\left(\theta_1 + \theta_2\right) \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) \\ -m_2 \left(g + l_1 \cos\left(\theta_1\right) \dot{\theta}_1^2 + l_{c2} \cos\left(\theta_1 + \theta_2\right) \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2\right) \\ \tau_2 - m_2 l_{c2} \left(g \sin\left(\theta_1 + \theta_2\right) + l_1 \sin\left(\theta_2\right) \dot{\theta}_1^2\right) \end{bmatrix}.$$

Using the constants given in the problem statement and the particular state of the mechanism, we get the solution $(\ddot{\theta}_1, \ddot{\theta}_2) = (-9.55276, -7.09408)\ \frac{\text{rad}}{\text{s}^2}$.

This result should be the same for both Lagrange's and Newton-Euler formulation since they are both accurate representations of the physical motion of the mechanism.