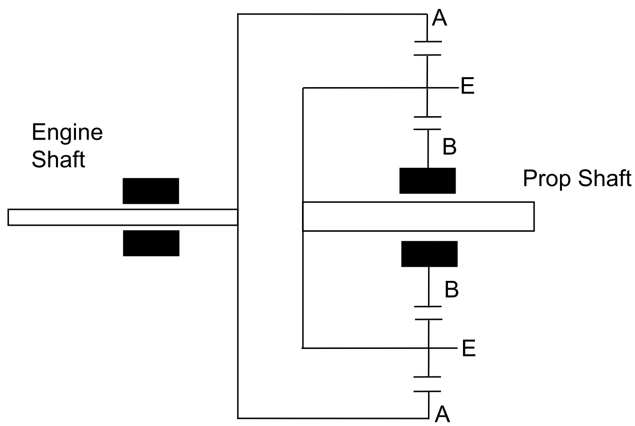
ME 380
Kinematics and Machine Dynamics
Homework VII Solutions

1. [25 points] The following skeleton diagram illustrates a planetary speed reduction system for an airplane propeller. The solid black boxes denote the ground, i.e., gear B is fixed to the ground. Find the propeller speed if $N_A = 193$, $N_E = 25$, $N_B = 143$, and the engine shaft speed is 4000 rpm.



Solution: Matching velocities at the pitch points of the meshing gears, we have

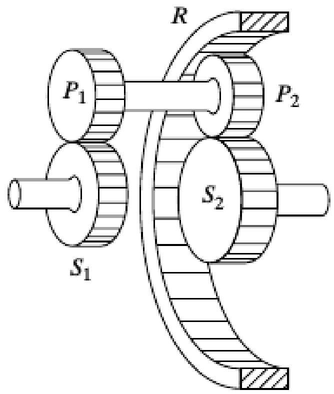
$$\omega_B r_B = \omega_P (r_B + r_E) - \omega_E r_E, \tag{1a}$$

$$\omega_A r_A = \omega_P (r_B + r_E) + \omega_E r_E. \tag{1b}$$

Gear B is fixed to the ground, so we set $\omega_B = 0$, solve for $\omega_E r_E$ from equation (1a) and substitute in (1b) to get

$$\omega_P = \frac{r_A}{r_A + r_B}\omega_A = \frac{N_A}{N_A + N_B}\omega_A = \frac{193}{193 + 143}4000 = 2297.62\,\text{rpm}.$$

2. [25 points] In the planetary train shown below, the input gear is sun S_1 and the output gear is sun S_2. Ring gear R is fixed. Planets P_1 and P_2 rotate at the same speed. Find the output-to-input speed ratio $\frac{\omega_{S_2}}{\omega_{S_1}}$ (in terms of tooth numbers N_{S_1}, etc.).



Solution: Matching velocities at the pitch points of the meshing gears, we have

$$\omega_C\left(r_{S_2}+r_{P_2}\right)-\omega_P r_{P_1} = \omega_{S_1} r_{S_1}, \tag{2a}$$

$$\omega_C\left(r_{S_2}+r_{P_2}\right)-\omega_P r_{P_2} = \omega_{S_2} r_{S_2}, \tag{2b}$$

$$\omega_C\left(r_{S_2}+r_{P_2}\right)+\omega_P r_{P_2} = \omega_R r_R = 0, \tag{2c}$$

where ω_C is the angular velocity of the carrier (arm). Plugging the solution of $\omega_C\left(r_{S_2}+r_{P_2}\right)$ from equation (2a) into equation (2b) and equation (2c), respectively, we get the following relationships

$$\omega_P\left(r_{P_1}-r_{P_2}\right)=\omega_{S_2} r_{S_2}-\omega_{S_1} r_{S_1},$$

$$\omega_P\left(r_{P_1}+r_{P_2}\right)=-\omega_{S_1} r_{S_1}.$$

Dividing the first equation by the second equation, we get

$$\frac{\omega_{S_2}}{\omega_{S_1}} = \frac{r_{S_1}}{r_{S_2}}\left(1-\frac{r_{P_1}-r_{P_2}}{r_{P_1}+r_{P_2}}\right)=\frac{N_{S_1}}{N_{S_2}}\left(1-\frac{N_{P_1}-N_{P_2}}{N_{P_1}+N_{P_2}}\right) \tag{3}$$

Note that, we have the relationships

$$r_R = r_{S_1}+r_{P_1}+r_{P_2} \iff N_R = N_{S_1}+N_{P_1}+N_{P_2},$$

$$r_{S_1}+r_{P_1} = r_{S_2}+r_{P_2} \iff N_{S_1}+N_{P_1}=N_{S_2}+N_{P_2}.$$

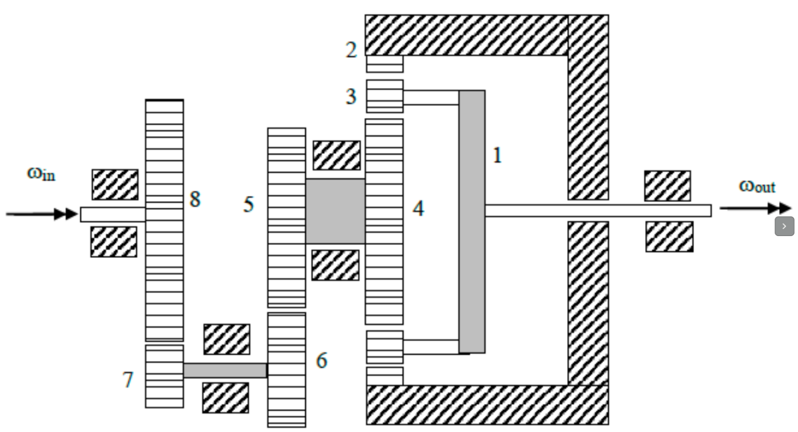
With these relationships, we can show that the solution (3) can alternatively be written as

$$\frac{\omega_{S_2}}{\omega_{S_1}} = \frac{1 + \frac{N_R}{N_{S_2}}}{1 + \frac{N_R N_{P_1}}{N_{S_1} N_{P_2}}}. \tag{4}$$

Indeed, the derivation proceeds as follows

$$\begin{aligned}
\frac{N_{S_1}}{N_{S_2}}\left(1 - \frac{N_{P_1} - N_{P_2}}{N_{P_1} + N_{P_2}}\right) &= \frac{N_{S_1}}{N_{S_2}}\left(\frac{2N_{P_2}}{N_{P_1} + N_{P_2}}\right)\cancelto{1}{\frac{N_{P_1} + N_{S_1}}{N_{P_1} + N_{S_1}}} \\
&= \frac{N_{S_1}}{N_{S_2}} \frac{2N_{P_2}\left(N_{P_2} + N_{S_2}\right)}{N_{P_2}N_{S_1} + N_{P_1}\left(N_{P_1} + N_{P_2} + N_{S_1}\right)} \\
&= \frac{N_{S_1}}{N_{S_2}} \frac{N_{P_2}\left(N_{S_2} + N_R\right)}{N_{P_2}N_{S_1} + N_{P_1}N_R} \\
&= \frac{\frac{N_{S_2}+N_R}{N_{S_2}}}{\frac{N_{P_2}N_{S_1}+N_{P_1}N_R}{N_{P_2}N_{S_1}}} \\
&= \frac{1 + \frac{N_R}{N_{S_2}}}{1 + \frac{N_{P_1}N_R}{N_{P_2}N_{S_1}}}.
\end{aligned}$$

3. [25 points] Gear 2 in the system shown below is fixed to the ground. Note that gears 6 and 7 are on the same shaft, as are gears 5 and 4. Use the following tooth numbers: $N_2 = 90$, $N_3 = 15$, $N_4 = 60$, $N_5 = 50$, $N_6 = 40$, $N_7 = 20$, $N_8 = 80$.
   (a) [10 points] Find the ratio $\frac{\omega_5}{\omega_8}$.
   (b) [15 points] Find the ratio $\frac{\omega_{\text{out}}}{\omega_{\text{in}}}$.



Solution: Matching velocities at the pitch points of the meshing gears, we have

$$\omega_8 r_8 = -\omega_7 r_7, \tag{5a}$$

$$\omega_6 r_6 = -\omega_5 r_5, \tag{5b}$$

$$\omega_1(r_3 + r_4) - \omega_3 r_3 = \omega_4 r_4, \tag{5c}$$

$$\omega_1(r_3 + r_4) + \omega_3 r_3 = \omega_2 r_2 = 0. \tag{5d}$$

Noting that $\omega_6 = \omega_7$ and dividing equation (5a) by equation (5b), we find

$$\frac{\omega_5}{\omega_8} = \frac{r_6}{r_5}\frac{r_8}{r_7} = \frac{N_6}{N_5}\frac{N_8}{N_7} = \frac{16}{5}. \tag{6}$$

Next, we solve equation (5d) for $\omega_3 r_3$ and substitute it in equation (5c), noting that $r_3 + r_4 = r_2 - r_3$

$$\omega_1(r_2 + r_4) = \omega_4 r_4 = \omega_5 r_4 \implies \omega_1 = \omega_5 \frac{N_4}{N_2 + N_4},$$

Substituting further from equation (6) for ω_5,

$$\frac{\omega_1}{\omega_8} = \frac{N_4 N_6 N_8}{(N_2 + N_4) N_5 N_7} = \frac{60 \cdot 40 \cdot 80}{(90 + 60) \cdot 50 \cdot 20} = \frac{32}{25} \tag{7}$$

4. [25 points] Design a double-dwell displacement function, s, that meets the requirements of the following DRDF cam timing description

- Dwell for $\frac{2\pi}{3}$ rad at 0 m (low dwell),
- Rise to $\frac{15}{100}$ m in $\frac{\pi}{3}$ rad,
- Dwell for $\frac{\pi}{3}$ rad at $\frac{15}{100}$ m (high dwell),
- Fall to 0 m in $\frac{2\pi}{3}$ rad.

Turn in your derivation of the polynomial s as well as your SVAJ diagram.

Solution: Define $h = \frac{15}{100}$, $\beta = \frac{\pi}{3}$ and $x = \frac{\theta}{\beta}$. For the rise phase, we have the following boundary conditions

$$\begin{aligned} s(2) &= 0, \quad s(3) = h, \\ v(2) &= 0, \quad v(3) = 0, \\ a(2) &= 0, \quad a(3) = 0. \end{aligned}$$

Since there are 6 boundary conditions, a fifth-order cam polynomial function suffices to satisfy the fundamental law of cam design, i.e. $s(x) = \sum_{k=0}^{5} \alpha_k x^k$. We have

$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 0 & 1 & 4 & 12 & 32 & 80 \\ 0 & 1 & 6 & 27 & 108 & 405 \\ 0 & 0 & 2 & 12 & 48 & 160 \\ 0 & 0 & 2 & 18 & 108 & 540 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

with the solution $\alpha = \begin{bmatrix} -76.8 & 162 & -135 & 55.5 & -11.25 & 0.9 \end{bmatrix}^\top$. For the fall phase, we have the following boundary conditions

$$\begin{aligned} s(4) &= h, \quad s(6) = 0, \\ v(4) &= 0, \quad v(6) = 0, \\ a(4) &= 0, \quad a(6) = 0. \end{aligned}$$

Again, there are 6 boundary conditions so a fifth-order cam polynomial function suffices to satisfy the fundamental law of cam design, i.e. $s(x) = \sum_{k=0}^{5} \beta_k x^k$. We have

$$\begin{bmatrix} 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \\ 0 & 1 & 8 & 48 & 256 & 1280 \\ 0 & 1 & 12 & 108 & 864 & 6480 \\ 0 & 0 & 2 & 24 & 192 & 1280 \\ 0 & 0 & 2 & 36 & 432 & 4320 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} = \begin{bmatrix} h \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

with the solution $\beta = \begin{bmatrix} 76.95 & -81 & 33.75 & -6.9375 & 0.703125 & -0.028125 \end{bmatrix}^\top$.

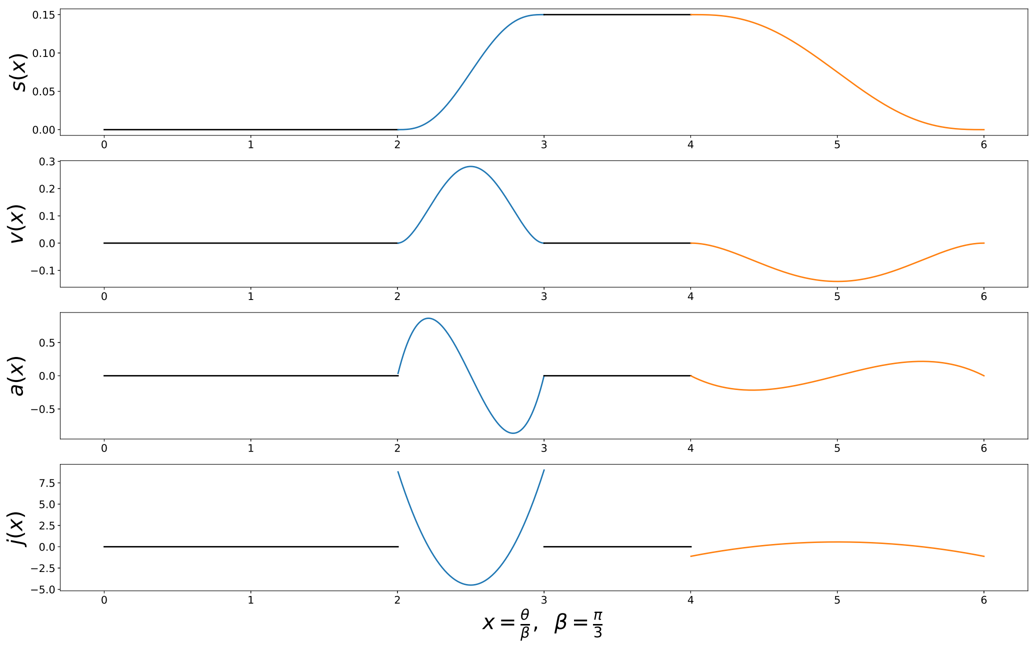


Figure 1: `S-V-A-J Diagram`

Rise polynomial

1. Normalization: let $x = \dfrac{\theta - 2\beta}{\beta}$, where $\beta = \pi/3$.

2. Boundary conditions: $s_r(0) = v_r(0) = a_r(0) = 0$
$s_r(1) = 3/20$, $v_r(1) = a_r(1) = 0$.

3. Polynomial selection: $s_r(x) = \sum_{n=0}^{5} \alpha_n x^n$

4. Matching the coefficients:

$$\alpha_0 = 0 \qquad \alpha_1 = 0 \qquad \alpha_2 = 0,$$
$$\sum_{n=0}^{5} \alpha_n = 3/20 \qquad \sum_{n=1}^{5} n\alpha_n = 0 \qquad \sum_{n=2}^{5} n(n-1)\alpha_n = 0,$$

which yields the linear system $A\alpha = b$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 3/20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Solving gives

$$\alpha = (\alpha_0 \ \cdots \ \alpha_5) = (0 \ \ 0 \ \ 0 \ \ 3/2 \ \ -9/4 \ \ 9/10).$$

6. Convert back to θ:

$$s_r(x)\Big|_{x=\frac{\theta-2\beta}{\beta}} = \frac{3}{2}\left(\frac{\theta-2\pi/3}{\pi/3}\right)^3 - \frac{9}{4}\left(\frac{\theta-2\pi/3}{\pi/3}\right)^4 + \frac{9}{10}\left(\frac{\theta-2\pi/3}{\pi/3}\right)^5$$

$$\boxed{s_r(\theta) = -\frac{384}{5} + \frac{486}{\pi}\theta - \frac{1215}{\pi^2}\theta^2 + \frac{2997}{2\pi^3}\theta^3 - \frac{3645}{4\pi^4}\theta^4 + \frac{2187}{10\pi^5}\theta^5.}$$

Fall polynomial

1. Normalization: let $x = \dfrac{\theta - 4\beta}{2\beta}$, where $\beta = \pi/3$.

2. Boundary conditions: $s_f(0) = 3/20$, $v_f(0) = a_f(0) = 0$
$s_f(1) = v_f(1) = a_f(1) = 0$.

3. Polynomial selection: $s_f(x) = \sum_{n=0}^{5} {}_n x^n$

4. Matching the coefficients:

$$\gamma_0 = 3/20 \qquad \gamma_1 = 0 \qquad \gamma_2 = 0,$$

$$\sum_{n=0}^{5}\gamma_n = 0 \qquad \sum_{n=1}^{5} n\gamma_n = 0 \qquad \sum_{n=2}^{5} n(n-1)\gamma_n = 0,$$

which yields the linear system $A\gamma = b$, where A is the same as before and

$$b = \begin{bmatrix} 3/20 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

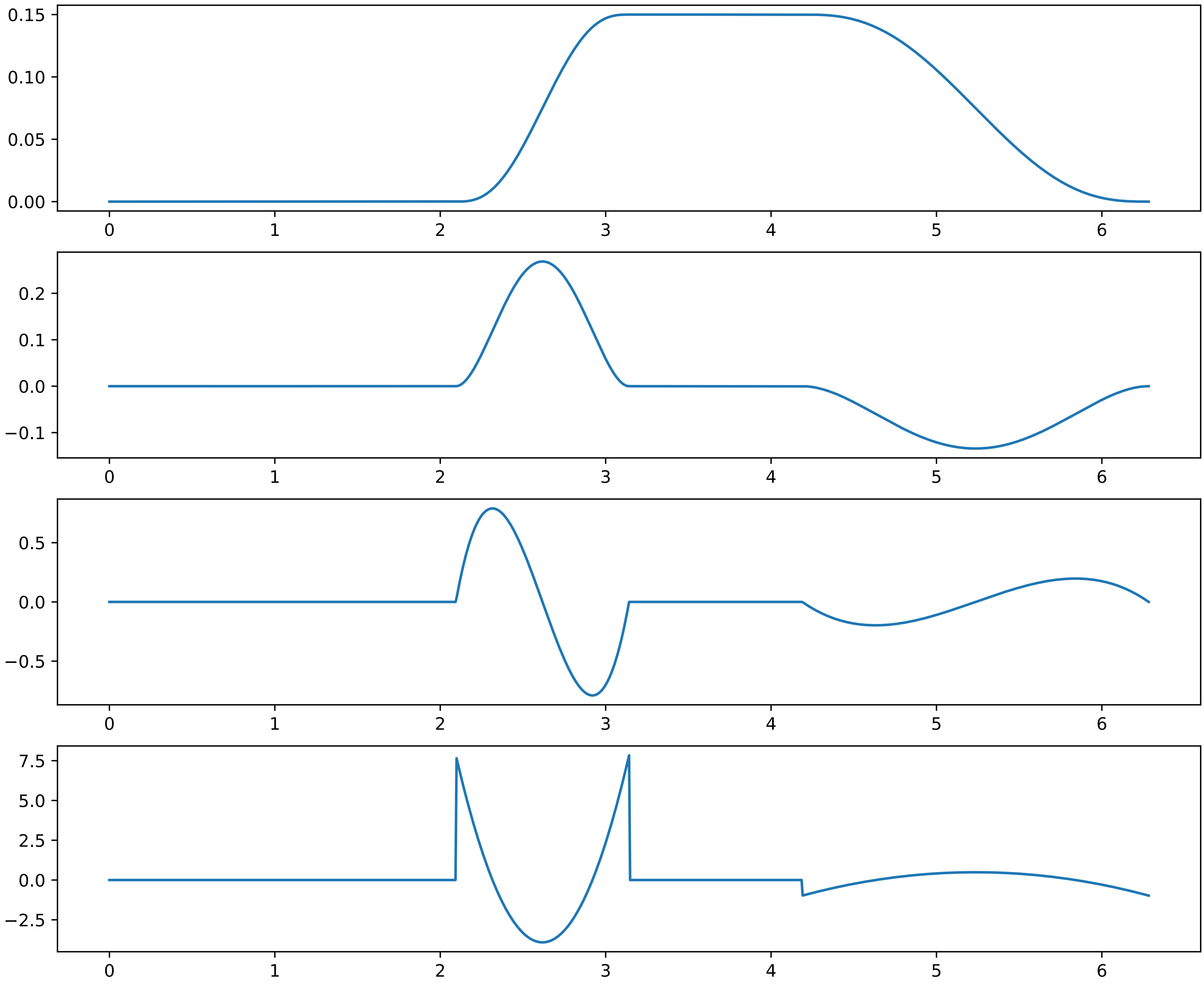
5. Solving gives

$$\gamma = \begin{bmatrix} \gamma_0 & \cdots & \gamma_5 \end{bmatrix} = \begin{bmatrix} 3/20 & 0 & 0 & -3/2 & 9/4 & -9/10 \end{bmatrix}$$

6. Convert back to θ:

$$s_f(x)\Big|_{x=\frac{\theta-2\beta}{\beta}} = \frac{3}{20} - \frac{3}{2}\left(\frac{\theta - 4\pi/3}{2\pi/3}\right)^3 + \frac{9}{4}\left(\frac{\theta - 4\pi/3}{2\pi/3}\right)^4 - \frac{9}{10}\left(\frac{\theta - 4\pi/3}{2\pi/3}\right)^5$$

$$\boxed{s_f(\theta) = \frac{1539}{20} - \frac{243}{\pi}\theta + \frac{1215}{4\pi^2}\theta^2 - \frac{2997}{16\pi^3}\theta^3 + \frac{3645}{64\pi^4}\theta^4 - \frac{2187}{320\pi^5}\theta^5.}$$


0.15
0.10
0.05
0.00
0
1
2
3
4
5
6
0.2
0.1
0.0
−0.1
0.5
0.0
−0.5
7.5
5.0
2.5
0.0
−2.5