6 Questions

If $a = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix}$ is a 3-vector, we define the skew-symmetrix matrix $S(a)$ as

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

1. [10 points] Verify, by direct calculation, that the operator S is linear; that is,

$$S(\alpha a + \beta b) = \alpha S(a) + \beta S(b),$$

for any vectors a and b belonging to $\mathbb{R}^3$ and scalars α and β.

2. [10 points] Verify, by direct calculation, that for any vectors a and p belonging to $\mathbb{R}^3$,

$$S(a)p = a \times p$$

where $a \times p$ denotes the vector cross product.

3. [10 points] Let $i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$, $j = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$, and $k = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. Verify, by direct calculation, that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} R_{x,\theta} = R_{x,\theta} S(i), \qquad \frac{\mathrm{d}}{\mathrm{d}\theta} R_{y,\theta} = R_{y,\theta} S(j), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\theta} R_{z,\theta} = R_{z,\theta} S(k)$$

4. [20 points] Compute the angular velocity of the second link of the Furuta pendulum, shown in Figure 1, relative to the world and express it in the world frame.

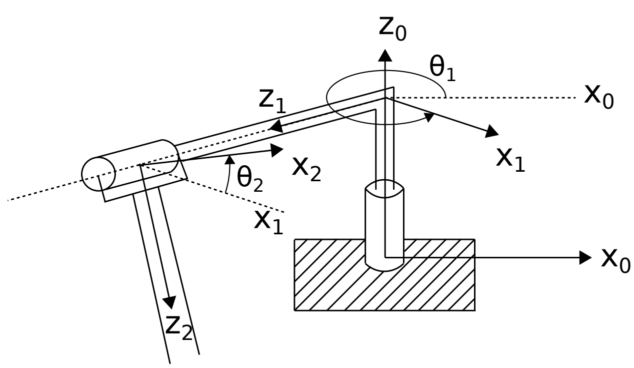

Figure 1: `Furuta Pendulum`

5. [10 points (bonus)] Given any square matrix A, the exponential of A is a matrix defined as

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \cdots.$$

Given a skew symmetrix matrix $S \in \mathfrak{so}(3)$, show that $SO(3) \ni e^S$ is a rotation matrix.

Hint: Use the fact that $e^A e^B = e^{A+B}$ provided that A and B commute, that is, $AB = BA$, and the fact that $\left| e^A \right| = e^{\operatorname{tr} A}$.

(1)
$$\alpha a + \beta b = \begin{bmatrix} \alpha a_x + \beta b_x \\ \alpha a_y + \beta b_y \\ \alpha a_z + \beta b_z \end{bmatrix}$$

$$\Rightarrow S(\alpha a + \beta b) = \begin{bmatrix} 0 & -(\alpha a_z + \beta b_z) & \alpha a_y + \beta b_y \\ \alpha a_z + \beta b_z & 0 & -(\alpha a_x + \beta b_x) \\ -(\alpha a_y + \beta b_y) & \alpha a_x + \beta b_x & 0 \end{bmatrix} \qquad (*)$$

On the other hand,

$$\alpha S(a) + \beta S(b) = \alpha \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(\alpha a_z + \beta b_z) & \alpha a_y + \beta b_y \\ \alpha a_z + \beta b_z & 0 & -(\alpha a_x + \beta b_x) \\ -(\alpha a_y + \beta b_y) & \alpha a_x + \beta b_x & 0 \end{bmatrix},$$

which is equal to $(*)$.

(2)
$$S(a)p = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix} \qquad (**)$$

$$a \times p = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ p_x & p_y & p_z \end{vmatrix} = i\left(a_y p_z - a_z p_y\right) - j\left(a_x p_z - a_z p_x\right) + k\left(a_x p_y - a_y p_x\right),$$

which is equal to (**).

(3) $$R_{y,\theta}^T\left(\frac{d}{d\theta}R_{y,\theta}\right) = \begin{bmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{bmatrix} \dot{\theta} \begin{bmatrix} -s_\theta & 0 & c_\theta \\ 0 & 0 & 0 \\ -c_\theta & 0 & -s_\theta \end{bmatrix}$$

$$= \dot{\theta} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = S(j).$$

Similar computations for $S(i)$ and $S(k)$.

(4) We can use addition of angular velocities to good effect here: ${}^0\omega^2 = {}^0\omega^1 + {}^1\omega^2$. Now,

$$S({}^0\omega^1) = R_{01}^T \dot{R}_{01} = R_{x,-\pi/2} \overbrace{R_{z,-\theta_1} \dot{R}_{z,\theta_1}}^{\dot{\theta}_1 S(k)} R_{x,\pi/2},$$

so ${}^0\omega^1$, expressed in Σ_1 is given by

$$({}^0\omega^1)^1 = \dot{\theta}_1 R_{x,-\pi/2}\, k,$$

which, when expressed in Σ_0 becomes

$$({}^0\omega^1)^0 = \dot{\theta}_1 R_{z,\theta_1} R_{x,\pi/2} R_{x,-\pi/2} k = \dot{\theta}_1 R_{z,\theta_1} k = \dot{\theta}_1 k .$$

Similarly,

$$S({}^1\omega^2) = R_{12}^T \dot{R}_{12} = R_{x,-\pi/2} \overbrace{R_{z,-\theta_2} \dot{R}_{z,\theta_2}}^{\dot{\theta}_2 S(k)} R_{x,\pi/2} \, ,$$

so ${}^1\omega^2$, expressed in Σ_2 is given by

$$({}^1\omega^2)^2 = \dot{\theta}_2 R_{x,-\pi/2} k \, ,$$

which, when expressed in Σ_0 becomes:

$$({}^1\omega^2)^0 = R_{z,\theta_1} R_{x,\pi/2} R_{z,\theta_2} \cancel{R_{x,\pi/2}} \, \dot{\theta}_2 \cancel{R_{x,-\pi/2}} k$$

$$= \dot{\theta}_2 R_{z,\theta_1} R_{x,\pi/2} k = \dot{\theta}_2 R_{z,\theta_1} (-j)$$

$$= \dot{\theta}_2 \left(\sin(\theta_1) i - \cos(\theta_1) j \right)$$

Finally, we can add:

$$\boxed{\begin{aligned} ({}^0\omega^2)^0 &= ({}^0\omega^1)^0 + ({}^1\omega^2)^0 \\ &= \sin(\theta_1)\dot{\theta}_2 i - \cos(\theta_1)\dot{\theta}_2 j + \dot{\theta}_1 k \end{aligned}}$$

(5) Let us use the definition of the exponential to show that

$$(e^S)^T = e^{-S}.$$

$$(e^S)^T = \left(I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \cdots\right)^T$$

$$= I + S^T + \frac{1}{2}(S^T)^2 + \frac{1}{3!}(S^T)^3 + \cdots$$

$$= e^{S^T} = e^{-S}.$$

Therefore,

$$e^S (e^S)^T = e^S e^{-S} \overset{\text{b/c } S \text{ and } S \text{ commute}}{=} e^{S-S} = e^0 = I.$$

Also, by using the hint,

$$|e^S| = e^{\operatorname{tr}(S)} = e^0 = 1.$$

Hence e^S is a rotation matrix.