Ancillary Material #3 Solutions

(1) Define the following vectors

p_0 : pos. vector from O_2 to O_1,

p_1 : " " " O_1 to B_1,

p_2 : " " " O_2 to B_2,

p_3 : " " " B_2 to B_1.

In terms of these quantities, the loop equation can be written as:

$$p_0 + p_1 - p_2 - p_3 = 0. \qquad (*)$$

$$p_0 = l_0 x_0, \quad p_1 = l_1 x_1, \quad p_2 = l_2 x_2 \quad p_3 = 0 \cdot x_2$$

$$(*) \quad l_0 x_0 + l_1 x_1 - l_2 x_2 = 0.$$

$$\text{Now} \quad R_{01} = \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R_{02} = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(*) \quad \begin{pmatrix} l_0 \\ 0 \\ 0 \end{pmatrix} + R_{01} \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} - R_{02} \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} l_0 + l_1 c_1 - l_2 c_2 \\ l_1 s_1 - l_2 s_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving these equations simultaneously yields

$$\left.\begin{aligned} l_1 c_1 &= l_2 c_2 - l_0 \\ l_1 s_1 &= l_2 s_2 \end{aligned}\right\}$$

$$\begin{aligned} & l_1^2 c_1^2 = l_2^2 c_2^2 - 2 l_0 l_2 c_2 + l_0^2 \\ + \; & l_1^2 s_1^2 = l_2^2 s_2^2 \\ \hline & l_1^2 = l_2^2 - 2 l_0 l_2 c_2 + l_0^2 \end{aligned}$$

$$\Rightarrow l_0^2 - l_1^2 - (2 l_0 c_2) l_2 + l_2^2 = 0 \Rightarrow l_2 \in \{15.78, \; 80.82\}$$

$$\tan(\theta_1) = \frac{l_2 s_2}{l_2 c_2 - l_0} \Rightarrow \theta_2 \in \{0.640, \; 3.025\}$$
$$= \{36.70^\circ, \; 173.3^\circ\}$$

(2) Let u_1, u_2, u_3 be unit vectors along $r_1, r_2,$ and r_3, respectively.

By the cross product method, we have

$$l_1 = \frac{-r_3 \cdot (u_2 \times k)}{u_1 \cdot (u_2 \times k)}, \quad l_2 = \frac{-r_3 \cdot (u_1 \times k)}{u_2 \cdot (u_1 \times k)}, \qquad (**)$$

where $k = [0 \ \ 0 \ \ 1]^T$. We know u_1, u_2 because we know the ratio of the components of r_1, r_2:

$$u_1 = \frac{1}{\sqrt{5}}[1 \ \ 2 \ \ 0]^T, \quad u_2 = \frac{1}{\sqrt{5}}[2 \ \ 1 \ \ 0]^T.$$

Now, we calculate

$$u_1 \times k = \hat{u}_1 k = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 \times k = \hat{u}_2 k = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

Next, we have

$$-u_1 \cdot (u_2 \times k) = u_2 \cdot (u_1 \times k) = \frac{3}{5}$$

$$-r_3 \cdot (u_2 \times k) = 20\sqrt{5}$$

$$-r_3 \cdot (u_1 \times k) = -5\sqrt{5}$$

Plugging these in $(**)$, we have

$$l_1 = -\frac{100\sqrt{5}}{3}, \quad l_2 = -\frac{25\sqrt{5}}{3},$$

as a result,

$$r_1 = l_1 u_1 = -\frac{100}{3}\begin{pmatrix}1\\2\\0\end{pmatrix}; \quad r_2 = l_2 u_2 = -\frac{25}{3}\begin{pmatrix}2\\1\\0\end{pmatrix}.$$

(3) Define the following quantities

$p_0 = \overrightarrow{EO}$, $p_1 = \overrightarrow{OB}$, $p_2 = \overrightarrow{BC}$, $p_3 = \overrightarrow{CE}$, $p_4 = \overrightarrow{OD}$, $p_5 = \overrightarrow{DE}$.

In terms of these quantities, two loop equations can be written as

$$p_0 + p_1 + p_2 + p_3 = 0 \qquad (3a)$$

$$p_4 + p_5 + p_6 = 0 \qquad (3b)$$

Writing these out in terms of frames planted on links 1, 2, and 3, with the x_i-axes aligned along the centerlines of the links gives

$$-l_E x_0 + l_1 x_1 + l_2 x_2 + l_3 x_3 = 0,$$

$$l_F x_0 + l_D y_0 + l_{DE} x_1 - l_E x_0 = 0,$$

where $l_3 = l_{DE} - l_{CD}$.

Expressing these equations in frame Σ_0 gives

$$\begin{pmatrix} -l_E + l_1 c_1 + l_2 c_{12} + l_3 c_3 \\ l_1 s_1 + l_2 s_{12} + l_3 s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} l_F - l_E + l_{DE} c_3 \\ l_D + l_{DE} s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving these equations with Newton-Raphson iteration, we get

θ_2	θ_3	l_D	l_E
$-17.002°$	-37.389	2.729	8.578

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(4) Known

r_1

$l_2 = \frac{3}{2} l_1$

$u_0 = (-1 \;\; 0 \;\; 0)^T$ (unit vector along r_0)

Unknown

u_2 (unit vector along r_2)

l_0 (magnitude of r_0).

This corresponds to case 4 in the cross product method with $u = r_0$, $v = r_2$ and $w = r_1$. We have

$$r_0 = \left\{ - r_1 \cdot u_0 \mp \sqrt{l_2^2 - \left[r_1 \cdot (u_0 \times k) \right]^2} \right\} u_0$$

$$r_1 = - \left[r_1 \cdot (u_0 \times k) \right] (u_0 \times k) \pm \sqrt{l_2^2 - \left[r_1 \cdot (u_0 \times k) \right]^2} \, u_0 \, .$$

Plugging the numerical quantities in these expressions, we get

$$r_0 = l_1 \left(\cos\left(\frac{2\pi}{9}\right) \mp \sqrt{\frac{9}{4} - \sin\left(\frac{2\pi}{9}\right)^2} \right) u_0 = 2.121 \, l_1 u_0$$

$$r_2 = - l_1 \left(\sin\left(\frac{2\pi}{9}\right) j \pm \sqrt{\frac{9}{4} - \sin\left(\frac{2\pi}{9}\right)^2} \, u_0 \right) =$$

$$= l_1 \begin{bmatrix} 1.355 \\ -0.643 \\ 0 \end{bmatrix}$$