1 (a) Let $v^2 = Rv^1$ and $w^2 = Rw^1$, where R rotates the coordinate axes of Σ_1 to those of Σ_2. Now,

$$v^2 \cdot w^2 = (v^2)^T w^2 = (Rv^1)^T R w^1 = (v^1)^T R^T R w^1 = (v^1)^T w^1 = v^1 \cdot w^1.$$

(b) $\|v\|^2 = v \cdot v = v^T v = v^T R^T R v = (Rv)^T R v = Rv \cdot Rv = \|Rv\|^2.$

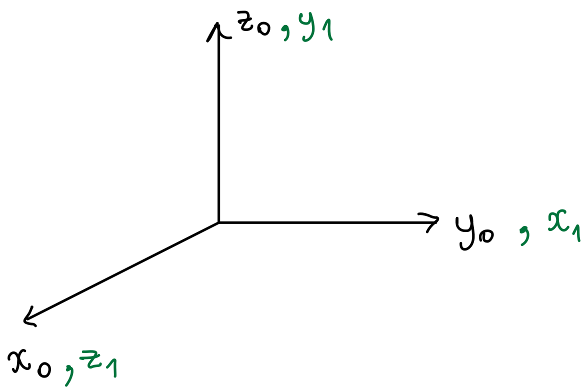
2 (a) $R^T R = I \Rightarrow r_i \cdot r_j = \delta_{ij}$

(b) $1 = |I| = |R^T R| = |R^T| \cdot |R| = |R|^2 \Rightarrow |R| = \pm 1.$

(c) For right-handed frames, we have $r_2 \times r_3 = r_1$ so that $r_1 \cdot r_2 \times r_3 = |R| = 1.$

3 $R_{01} = R_{x,\pi/2} R_{y,\pi/2}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



4

$$R_{23} = R_{12}^T R_{13}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \end{bmatrix}$$

5

$$R = R_{x,\pi/2} R_{y,0} R_{z,\pi/4}$$

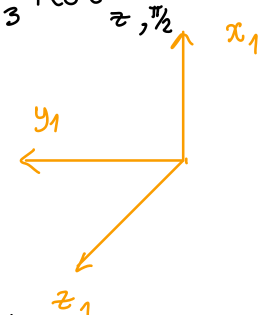
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & -1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix}$$

$$x_1^0 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T.$$

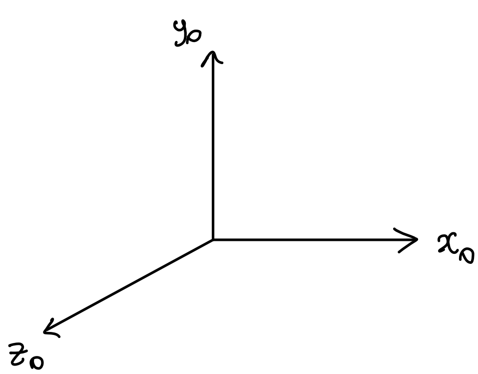
6

$$H = \text{Trans}_{y,1}\ \text{Trans}_{x,3}\ \text{Rot}_{z,\pi/2}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$o_1^0 = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix}^T.$$



7 (c) $H = \text{Trans}_{y,-1}\,\text{Rot}_{y,\frac{3\pi}{4}}\,\text{Rot}_{z,-\pi/8}\,\text{Trans}_{z,3/2}\,\text{Rot}_{x,\frac{2\pi}{3}}$

$$= \text{Rot}_{y,\frac{3\pi}{4}}\,\text{Trans}_{y,-1}\,\text{Trans}_{z,3/2}\,\text{Rot}_{z,-\pi/8}\,\text{Rot}_{x,\frac{2\pi}{3}}$$

$$= \begin{bmatrix} c_{3\pi/4} & 0 & s_{3\pi/4} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{3\pi/4} & 0 & c_{\frac{3\pi}{4}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ -1 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \begin{matrix} 0 \\ 0 \\ 3/2 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cdot$$

$$\cdot \begin{bmatrix} c_{-\pi/8} & -s_{-\pi/8} & 0 & 0 \\ s_{-\pi/8} & c_{-\pi/8} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\pi/3} & -s_{2\pi/3} & 0 \\ 0 & s_{2\pi/3} & c_{2\pi/3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 1 & 0 & -1 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2+\sqrt{2}} & -\frac{1}{2}\sqrt{2-\sqrt{2}} & 0 & 0 \\ \frac{1}{2}\sqrt{2-\sqrt{2}} & \frac{1}{2}\sqrt{2+\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(\pi/4) = \cos(\pi/8)\cos(\pi/8) - \sin(\pi/8)\sin(\pi/8) = \cos^2(\pi/8) - \sin^2(\pi/8)$$

$$= 2\cos^2(\pi/8) - 1 = 1 - 2\sin^2(\pi/8)$$

$$\Rightarrow \quad \cos(\pi/8) = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\Rightarrow \quad \sin(\pi/8) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

(Aside)

$$= \begin{bmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 1 & 0 & -1 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{4}\sqrt{2-\sqrt{2}} & \frac{\sqrt{3}}{4}\sqrt{2-\sqrt{2}} & 0 \\ -\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{1}{4}\sqrt{2+\sqrt{2}} & -\frac{\sqrt{3}}{4}\sqrt{2+\sqrt{2}} & 0 \\ 0 & \sqrt{3}/2 & -1/2 & 3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{8}\left(\sqrt{2}\sqrt{2-\sqrt{2}}+2\sqrt{6}\right) & \frac{1}{8}\left(\sqrt{6}\sqrt{2-\sqrt{2}}-2\sqrt{2}\right) & \frac{3\sqrt{2}}{4} \\ -\frac{1}{2}\sqrt{2-\sqrt{2}} & -\frac{1}{4}\sqrt{2+\sqrt{2}} & -\frac{\sqrt{3}}{4}\sqrt{2+\sqrt{2}} & -1 \\ -\frac{\sqrt{2}}{4}\sqrt{2+\sqrt{2}} & \frac{1}{8}\left(\sqrt{2}\sqrt{2-\sqrt{2}}-2\sqrt{6}\right) & \frac{1}{8}\left(\sqrt{6}\sqrt{2-\sqrt{2}}+2\sqrt{2}\right) & -\frac{3\sqrt{2}}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.653 & 0.748 & -0.119 & 1.061 \\ -0.383 & -0.462 & -0.800 & -1.000 \\ -0.653 & -0.477 & 0.588 & -1.061 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$