ME 380: Kinematics and Machine Dynamics
Rigid Body Motion: Rotations and Translations

1 How to use this assignment

In this assignment, students will be reminded basic facts about linear algebra. They will use these facts to define and manipulate coordinate frames in 3D-space.

- The instructor will provide questions and provide intermittent help.
- This activity provides the opportunity for the students to learn / recall important definitions and facts on linear algebra. They will use these tools to define and perform operations on coordinate frames.
- Students will spend approximately 30 minutes on this activity.
- This activity will be submitted and graded.

2 Learning objectives

After completing this assignment, students will be able to

- Students will be able to perform dot and cross multiplication, take inner products, compute norms, multiply and invert matrices.
- They will define coordinate frames and learn to express vectors in various such frames.

3 Assignment completion instructions

1. Planning
   - What will the instructor need before the activity?
   - What will the student need before the activity?
     - Pencil and paper
     - Calculator
2. Source material(s) for the activity
   - Lecture notes
   - Resources placed on Piazza.
3. Introduction and warm-up practice activity for the students
   - Introduction and instructions
     - A vector contains a magnitude and a direction.

  - A coordinate frame in 3D-space is comprised of three *orthonormal* vectors in $\mathbb{R}^3$.
  - A vector has different *components* in different coordinate frames.
- Warm-up and practice
  - Brainstorm
  - Think, pair, share:
    - We use the dot (inner) product to find components of a vector on another vector.
    - Matrix multiplication is not commutative: $AB \neq BA$ unless A and B are special.
    - A *rotation* matrix is an *orthogonal* matrix with unit *determinant*.

4. Main activity
   - Students can submit a soft-copy via email or hand-in a hard-copy per group.
   - This activity graded.
5. Wrap-up / reflection
   - Peer review
     - Have a neighboring group review your work.
     - Neighboring group should provide opinion about whether or not the solution and its development are correct.
     - After the peer review is done, the owner group tries to incorporate the suggestions and improve the solution.
   - 3-2-1 Countdown
   - One-minute paper

4 Conclusion and moving forward

This activity is aimed to teach the students the necessary linear algebra background to define coordinate frames and manipulate vectors in multiple such frames.

5 Grading criteria

- Preparation
- Critical thinking
- Technical expertise
- Adherence to assignment guidelines
- Timeliness

5.1 Peer reviewer notes

1. Check if various vector-vector, matrix-vector, or matrix-matrix multiplications are well-defined and correctly performed.

2. Check if the angle between two vectors are correctly computed.

3. Check if vectors are expressed correctly in different coordinate frames.

5.2 Preparation and content

Students can prepare for this assignment by reviewing the lecture notes. Specifically, students should review ...

5.3 Creativity and communication

Students should be able to explain their thinking process to a fellow student in order to make sure that they have learnt the material.

5.4 Critical thinking

Students should think about the fact that while points and vectors are inherent objects in a Euclidean space, their representations in different coordinate frames are additional constructs and depend on the particular coordinate frame.

5.5 Engagement with topic

Every student should be able to and attempt to explain how they think about the problem to a fellow student.

5.6 Technical expertise

Linear algebra

6 Questions

1. [10 points] This question explores the properties of the dot product.
   (a) [5 points] Using the fact that $v \cdot w = v^\top w$, show that the dot product of two vectors does not depend on the choice of frames in which their coordinates are defined.
   (b) [5 points] Use part (a) to show that the length of a vector is not changed by rotation, that is, $\|v\| = \|Rv\|$.

2. [15 points] If a matrix R satisfies $R^\top R = I$, show that the following are true.
   (a) [5 points] Column vectors of R are of unit length and mutually perpendicular.
   (b) [5 points] $|R| = \pm 1$.
   (c) [5 points] $|R| = +1$, if we restrict ourselves to right-handed coordinate frames.

3. [10 points] If the coordinate system Σ_1 is obtained from the coordinate system Σ_0 by a rotation of $\pi/2$ about the x-axis followed by a rotation of $\pi/2$ about the current y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

4. [10 points] Suppose that three coordinate frames Σ_1, Σ_2, and Σ_3 are given, and suppose

$$R_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}, \qquad R_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   Find the matrix R_{23}.

5. [10 points] Find the rotation matrix that corresponds to the EulerXYZ angles: $\phi = \pi/2$, $\theta = 0$, $\psi = \pi/4$. What is the direction of the x_1-axis relative to the base frame?

6. [15 points] Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\pi/2$ about the current z-axis followed by a translation of 1 unit along the fixed (world) y-axis. Sketch the frame. What are the coordinates of the origin o_1 with respect to the original frame in each case?

7. [25 points] (a) [10 points] Write three functions: `RotX`, `RotY`, `RotZ`, each of which takes an angle (in radians) argument and returns the corresponding basic homogeneous matrix that corresponds to a pure rotation about the relevant axis.
   (b) [10 points] Write three functions: `TransX`, `TransY`, `TransZ`, each of which takes a displacement and returns the corresponding basic homogeneous matrix that corresponds to a pure translation along the relevant axis.
   (c) [5 points] Use parts (a) and (b) to compute the homogeneous transformation that corresponds to the rigid-body motion that
      (i) Translates along the y-axis by -1 units,
      (ii) Rotates about the current y-axis by $3\pi/4$ radians,

(iii) Rotates about the current z-axis by $-\pi/8$ radians,
(iv) Translates along the current z-axis by $3/2$ units,
(v) Rotates about the current x-axis by $2\pi/3$ radians.